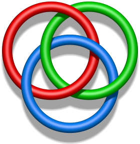
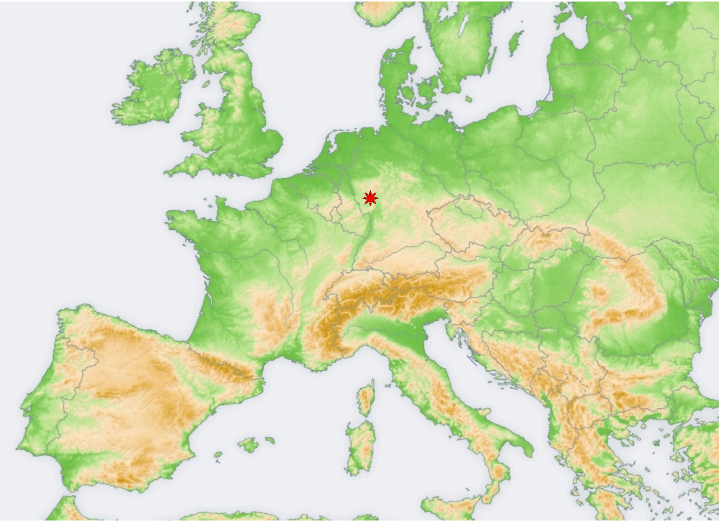
Multiparticle Quantum Entanglement

Otfried Gühne



Department Physik, Universität Siegen

Theoretical Quantum Optics



General Structure

1. Lecture I: Pure states
2. Lecture II: Mixed states
3. Lecture III: Graph states and other families of states

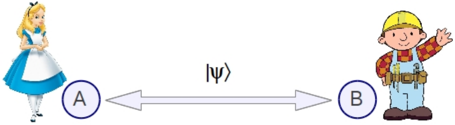
Schedule for Lecture I

1. Basic facts about bipartite entanglement
2. Three qubits: GHZ and W
3. Beyond three qubits
4. Quantifying multipartite entanglement

Bipartite Entanglement

Entanglement

Alice and Bob share a state $|\psi\rangle$.



Definition: A pure state $|\psi\rangle$ is separable iff it is a product state:

$$|\psi\rangle = |a\rangle_A |b\rangle_B = |a, b\rangle.$$

Otherwise it is called entangled.

Examples

Separable states:

$$|\phi_1\rangle = |01\rangle, \quad |\phi_2\rangle = |00\rangle + |01\rangle + |10\rangle + |11\rangle = (|0\rangle + |1\rangle)(|0\rangle + |1\rangle)$$

Entangled states

$$|\psi_1\rangle = |01\rangle + |10\rangle, \quad |\psi_2\rangle = |00\rangle + |01\rangle + |10\rangle - |11\rangle$$

Entanglement

Properties of entanglement

- Entanglement is invariant under local changes of the basis.
- Entanglement is necessary for a Bell inequality violation.
- Entanglement is a resource for teleportation & cryptography.

Bell states

Popular entangled states are the Bell states:

$$|\psi^-\rangle = |01\rangle - |10\rangle, \qquad |\psi^+\rangle = |01\rangle + |10\rangle,$$
$$|\phi^-\rangle = |00\rangle - |11\rangle, \qquad |\phi^+\rangle = |00\rangle + |11\rangle.$$

They are maximally entangled.

The Schmidt decomposition

Decomposition

For any bipartite state there are local bases for Alice and Bob such that

$$|\psi\rangle = \sum_{i=1}^{R} s_i |ii\rangle.$$

The s_i are the Schmidt coefficients, positive and unique. The number R is the Schmidt rank of the state.

The Schmidt decomposition

Decomposition

For any bipartite state there are local bases for Alice and Bob such that

$$|\psi\rangle = \sum_{i=1}^{R} s_i |ii\rangle.$$

The s_i are the Schmidt coefficients, positive and unique. The number R is the Schmidt rank of the state.

Maximally entangled states

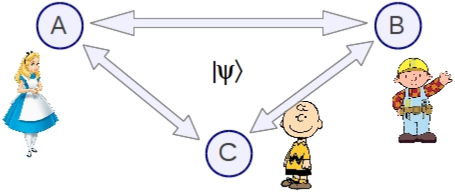
A bipartite state is maximally entangled, if the marginals are max. mixed

$$\varrho_A = Tr_B(|\psi\rangle\langle\psi|) = \frac{\mathbb{1}}{d} \quad \Leftrightarrow \quad s_i = \frac{1}{\sqrt{d}}$$

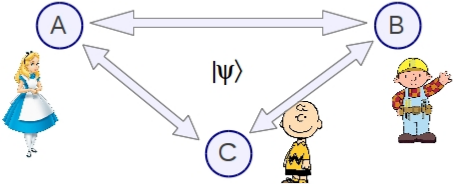
Examples: Bell states and $|\phi^+\rangle = (\sum_i |ii\rangle)/\sqrt{d}$.

Three qubits

Multiparticle entanglement



Multiparticle entanglement



There are different possibilities:

- Fully separable:

$$|\psi^{\text{fs}}\rangle = |000\rangle$$

- Biseparable:

$$|\psi^{\text{bs}}\rangle = |0\rangle \otimes (|00\rangle + |11\rangle)$$

- Genuine multiparticle entangled:

$$|GHZ\rangle = |000\rangle + |111\rangle \quad \text{or} \quad |W\rangle = |001\rangle + |010\rangle + |100\rangle$$

- $|GHZ\rangle$ and $|W\rangle$ are generalized Bell states. What's the difference?

Local unitary equivalence

Simple observation

GHZ and W state have different one-qubit marginals

$$\varrho_A^{GHZ} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|), \quad \varrho_A^{W} = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|.$$

They cannot be LU equivalent.

Local unitary equivalence

Simple observation

GHZ and W state have different one-qubit marginals

$$\varrho_A^{GHZ} = \frac{1}{2}(|0\rangle\langle 0| + |1\rangle\langle 1|), \quad \varrho_A^{W} = \frac{2}{3}|0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1|.$$

They cannot be LU equivalent.

General fact

There is no Schmidt decomposition for multiparticle systems.
But: Any three-qubit state can be simplified via LU to

$$|\psi\rangle = \lambda_0|000\rangle + \lambda_1 e^{i\theta}|100\rangle + \lambda_2|101\rangle + \lambda_3|110\rangle + \lambda_4|111\rangle,$$

with $\lambda_i \geq 0$ and $0 \leq \theta \leq \pi$.

A. Peres, PLA 202, 16 (1995); A. Acin et al., PRL 85, 1560 (2000).

LOCC

Local operations and classical communication

LOCC transformations are a sequence of steps where each party either

- adds an ancilla quantum system,
- applies a local unitary transformation,
- performs a measurement and communicates the result.

LOCC protocols may require infinitely many rounds ...

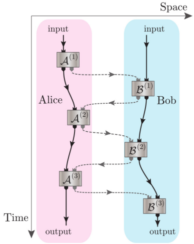


Figure: S. Akibue et al., PRA 96, 062331 (2017).

LOCC

Local operations and classical communication

LOCC transformations are a sequence of steps where each party either

- adds an ancilla quantum system,
- applies a local unitary transformation,
- performs a measurement and communicates the result.

LOCC protocols may require infinitely many rounds ...

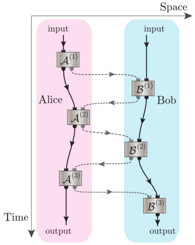


Figure: S. Akibue et al., PRA 96, 062331 (2017).

- Nielsen's theorem: For bipartite states $|\psi\rangle \xrightarrow{LOCC} |\phi\rangle$ is possible iff

$$\vec{s}(\psi) \prec \vec{s}(\phi) \Leftrightarrow s_1(\psi) \leq s_1(\phi),\ s_1(\psi) + s_2(\psi) \leq s_1(\phi) + s_2(\phi) ...$$

- For multipartite systems, LOCC orbits are difficult ...

M.A. Nielsen, PRL 83, 436 (1999), J.I. de Vicente et al., PRL 111, 110502 (2013).

SLOCC

Stochastic LOCC

Given a single copy of $|\psi\rangle$ can we achieve with some probability $p > 0$

$$|\psi\rangle \xrightarrow{SLOCC} |\phi\rangle$$

with LOCC?

SLOCC

Stochastic LOCC

Given a single copy of $|\psi\rangle$ can we achieve with some probability $p > 0$

$$|\psi\rangle \stackrel{SLOCC}{\longrightarrow} |\phi\rangle$$

with LOCC?

Mathematical formulation

One can reach $|\psi\rangle \stackrel{SLOCC}{\longrightarrow} |\phi\rangle$ iff there are matrices A, B, and C such that

$$|\phi\rangle = A \otimes B \otimes C|\psi\rangle.$$

If the matrices are invertible, then $|\psi\rangle$ and $|\phi\rangle$ are SLOCC equivalent.

W. Dür, G. Vidal, J.I. Cirac, PRA 62, 062314 (2000).

Equivalence classes for three qubits

Result

One cannot transform

$$|GHZ\rangle = |000\rangle + |111\rangle \overset{SLOCC}{\not\leftrightarrow} |W\rangle = |001\rangle + |010\rangle + |100\rangle$$

Any other fully entangled state can be transformed into $|GHZ\rangle$ or $|W\rangle$.

W. Dür, G. Vidal, J.I. Cirac, PRA 62, 062314 (2000).

Equivalence classes for three qubits

Result

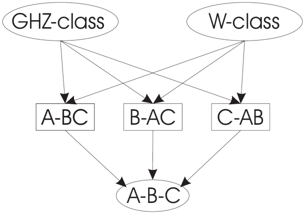
One cannot transform

$$|GHZ\rangle = |000\rangle + |111\rangle \overset{SLOCC}{\not\longleftrightarrow} |W\rangle = |001\rangle + |010\rangle + |100\rangle$$

Any other fully entangled state can be transformed into $|GHZ\rangle$ or $|W\rangle$.

W. Dür, G. Vidal, J.I. Cirac, PRA 62, 062314 (2000).

Six classes of entanglement



For more that three qubits there are infinitely many equivalence classes.

F. Verstraete, J. Dehaene, B. De Moor, PRA 68, 012103 (2003).

Mathematical generalization: Tensor rank

Mathematical generalization: Tensor rank

Observation

The rank R of a matrix M_{ij} can be defined via the minimal decomposition into rank-one matrices,

$$M_{ij} = \sum_{r=1}^{R} a_i^{(r)} b_j^{(r)}.$$

This can be generalized to the tensor rank,

$$T_{ijk} = \sum_{r=1}^{R} a_i^{(r)} b_j^{(r)} c_k^{(r)}$$

Mathematical generalization: Tensor rank

Observation

The rank R of a matrix M_{ij} can be defined via the minimal decomposition into rank-one matrices,

$$M_{ij} = \sum_{r=1}^{R} a_i^{(r)} b_j^{(r)}.$$

This can be generalized to the tensor rank,

$$T_{ijk} = \sum_{r=1}^{R} a_i^{(r)} b_j^{(r)} c_k^{(r)}$$

- Tensor rank over $\mathbb{R}$ and $\mathbb{C}$ differ.
- Matrix multiplication can be formalized using a specific tensor. Knowing its rank simplifies algorithms.
  V. Strassen, J. Reine Angewandte Math. 264, 184 (1973).
- The tensor rank is very difficult to calculate.
  A. Fawzi et al., Nature 610, 47 (2022).

Application to quantum states

- For pure states, tensor rank asks for the minimal decomposition

$$|\psi\rangle = \sum_{r=1}^{R} |a^{(r)}\rangle |b^{(r)}\rangle |c^{(r)}\rangle$$

  into (not necessarily orthonormal) product vectors.
- This is invariant under invertible SLOCC, $|\psi\rangle \mapsto A \otimes B \otimes C|\psi\rangle$.

Application to quantum states

- For pure states, tensor rank asks for the minimal decomposition

$$|\psi\rangle = \sum_{r=1}^{R} |a^{(r)}\rangle |b^{(r)}\rangle |c^{(r)}\rangle$$

  into (not necessarily orthonormal) product vectors.
- This is invariant under invertible SLOCC, $|\psi\rangle \mapsto A \otimes B \otimes C|\psi\rangle$.
- The GHZ state has tensor rank 2, the W state tensor rank 3.
- The tensor rank can be seen as a quantifier of entanglement, the Schmidt measure.

Application to quantum states

- For pure states, tensor rank asks for the minimal decomposition

$$|\psi\rangle = \sum_{r=1}^{R} |a^{(r)}\rangle|b^{(r)}\rangle|c^{(r)}\rangle$$

  into (not necessarily orthonormal) product vectors.
- This is invariant under invertible SLOCC, $|\psi\rangle \mapsto A \otimes B \otimes C|\psi\rangle$.
- The GHZ state has tensor rank 2, the W state tensor rank 3.
- The tensor rank can be seen as a quantifier of entanglement, the Schmidt measure.
- Interestingly,

$$|GHZ\rangle = |000\rangle + |111\rangle \overset{SLOCC}{\sim} (|0\rangle + \varepsilon|1\rangle)^{\otimes 3} - |000\rangle \xrightarrow{\varepsilon \to 0} |W\rangle$$

  $\Rightarrow$ W states form a set of measure zero [$\theta = \lambda_4 = 0$ in SLD].

J. Eisert, H.-J. Briegel, PRA 64, 022306 (2001)

Back to physics: Bell inequalities for the GHZ state

Back to physics: Bell inequalities for the GHZ state

- Consider three particles with the measurements $X_1, Y_1, ..., X_3, Y_3$ and the measurement results ± 1.
- What is

$$X_1 X_2 X_3 - Y_1 Y_2 X_3 - X_1 Y_2 Y_3 - Y_1 X_2 Y_3 = ?$$

Back to physics: Bell inequalities for the GHZ state

- Consider three particles with the measurements $X_1, Y_1, ..., X_3, Y_3$ and the measurement results ± 1.
- What is

$$X_1 X_2 X_3 - Y_1 Y_2 X_3 - X_1 Y_2 Y_3 - Y_1 X_2 Y_3 = ?$$

- We have $(Y_1 Y_2 X_3) \times (X_1 Y_2 Y_3) \times (Y_1 X_2 Y_3) = X_1 X_2 X_3$ $\Rightarrow (Y_1 Y_2 X_3) = (X_1 Y_2 Y_3) = (Y_1 X_2 Y_3) = -1$ and $X_1 X_2 X_3 = 1$ is impossible.

Back to physics: Bell inequalities for the GHZ state

- Consider three particles with the measurements $X_1, Y_1, ..., X_3, Y_3$ and the measurement results ± 1.
- What is

$$X_1 X_2 X_3 - Y_1 Y_2 X_3 - X_1 Y_2 Y_3 - Y_1 X_2 Y_3 = ?$$

- We have $(Y_1 Y_2 X_3) \times (X_1 Y_2 Y_3) \times (Y_1 X_2 Y_3) = X_1 X_2 X_3$ $\Rightarrow (Y_1 Y_2 X_3) = (X_1 Y_2 Y_3) = (Y_1 X_2 Y_3) = -1$ and $X_1 X_2 X_3 = 1$ is impossible.

Mermin inequality

So

$$X_1 X_2 X_3 - Y_1 Y_2 X_3 - X_1 Y_2 Y_3 - Y_1 X_2 Y_3 \leq 2.$$

D. Greenberger, M. Horne, A. Zeilinger, arXiv:0712.0921, D. Mermin, PRL 1991.

The GHZ argument

- Consider then the GHZ state

$$|GHZ\rangle = |000\rangle + |111\rangle$$

and X, Y, Z are the Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};$$

- The GHZ state is an eigenstate

$$X_1 X_2 X_3 |GHZ\rangle = |GHZ\rangle; \quad Z_1 Z_2 \mathbb{1} |GHZ\rangle = |GHZ\rangle;$$
$$Z_1 \mathbb{1} Z_3 |GHZ\rangle = |GHZ\rangle; \quad \mathbb{1} Z_2 Z_3 |GHZ\rangle = |GHZ\rangle.$$

The GHZ argument

- So: $|GHZ\rangle$ is also an eigenstate of the products

$$(X_1X_2X_3) \times (Z_1Z_2\mathbb{1})|GHZ\rangle = (-Y_1Y_2X_3)|GHZ\rangle = |GHZ\rangle$$

The GHZ argument

- So: $|GHZ\rangle$ is also an eigenstate of the products

$$(X_1X_2X_3) \times (Z_1Z_2\mathbb{1})|GHZ\rangle = (-Y_1Y_2X_3)|GHZ\rangle = |GHZ\rangle$$

- Hence

$$\langle GHZ|(-Y_1Y_2X_3)|GHZ\rangle = 1$$

The GHZ argument

- So: $|GHZ\rangle$ is also an eigenstate of the products

$$(X_1X_2X_3) \times (Z_1Z_2\mathbb{1})|GHZ\rangle = (-Y_1Y_2X_3)|GHZ\rangle = |GHZ\rangle$$

- Hence

$$\langle GHZ|(-Y_1Y_2X_3)|GHZ\rangle = 1$$

- Finally

$$\langle X_1X_2X_3\rangle - \langle Y_1Y_2X_3\rangle - \langle X_1Y_2Y_3\rangle - \langle Y_1X_2Y_3\rangle = 4.$$

The GHZ argument

- So: $|GHZ\rangle$ is also an eigenstate of the products

$$(X_1X_2X_3) \times (Z_1Z_2\mathbb{1})|GHZ\rangle = (-Y_1Y_2X_3)|GHZ\rangle = |GHZ\rangle$$

- Hence

$$\langle GHZ|(-Y_1Y_2X_3)|GHZ\rangle = 1$$

- Finally

$$\langle X_1X_2X_3\rangle - \langle Y_1Y_2X_3\rangle - \langle X_1Y_2Y_3\rangle - \langle Y_1X_2Y_3\rangle = 4.$$

Where is the mistake?

The GHZ argument

- So: $|GHZ\rangle$ is also an eigenstate of the products

$$(X_1 X_2 X_3) \times (Z_1 Z_2 \mathbb{1})|GHZ\rangle = (-Y_1 Y_2 X_3)|GHZ\rangle = |GHZ\rangle$$

- Hence

$$\langle GHZ|(-Y_1 Y_2 X_3)|GHZ\rangle = 1$$

- Finally

$$\langle X_1 X_2 X_3\rangle - \langle Y_1 Y_2 X_3\rangle - \langle X_1 Y_2 Y_3\rangle - \langle Y_1 X_2 Y_3\rangle = 4.$$

Where is the mistake?

Note: Experimentally, one finds

$$\langle X_1 X_2 X_3\rangle - \langle Y_1 Y_2 X_3\rangle - \langle X_1 Y_2 Y_3\rangle - \langle Y_1 X_2 Y_3\rangle = 2.82.$$

J.W. Pan et al, Nature 1999, more recent experiments found values close to four.

Hidden assumptions

While proving $X_1X_2X_3 - Y_1Y_2X_3 - X_1Y_2Y_3 - Y_1X_2Y_3 \leq 2$ two assumptions have been made

- Measurements have values ± 1 independently of whether they are measured or not (realism)
- The value of X_1 does not depend on whether X_2 or Y_2 is measured (locality)

Hidden assumptions

While proving $X_1X_2X_3 - Y_1Y_2X_3 - X_1Y_2Y_3 - Y_1X_2Y_3 \leq 2$ two assumptions have been made

- Measurements have values ± 1 independently of whether they are measured or not (realism)
- The value of X_1 does not depend on whether X_2 or Y_2 is measured (locality)

One of these assumptions must be wrong

$\Rightarrow$ GHZ states are non-local in an extreme manner.

Properties of the W state

Properties of the W state

Entanglement in marginals

- For the GHZ state, the two-body marginal is separable.

$$\varrho_{AB} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11|)$$

- For the W state the marginal is entangled. In fact there is no state with more entanglement in the marginals.
- The W state is uniquely determined by the marginals, the GHZ state not.

What are the interesting multiqubit states?

- The GHZ states violate Bell inequalities maximally:

$$|GHZ\rangle = |0000\rangle + |1111\rangle$$

- The W-states are robust against qubit loss:

$$|W\rangle = |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle$$

- The cluster states are useful for the one-way quantum computer:

$$|CL\rangle = |0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle$$

- The Dicke states are often easy to prepare:

$$|D\rangle = |0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle$$

- The singlet states are $U \otimes ... \otimes U$ invariant:

$$|\psi^{(4)}\rangle = |0011\rangle + |1100\rangle - \frac{1}{2}(|10\rangle + |10\rangle) \otimes (|10\rangle + |10\rangle)$$

Beyond three qubits

Multipartite entanglement

Definition

A pure N-qubit state $|\psi\rangle$ is k-separable, if we can write

$$|\psi^{(n)}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes ... \otimes |\phi_k\rangle,$$

that is, the system can be divided into k uncorrelated parts.

Multipartite entanglement

Definition

A pure N-qubit state $|\psi\rangle$ is k-separable, if we can write

$$|\psi^{(n)}\rangle = |\phi_1\rangle \otimes |\phi_2\rangle \otimes ... \otimes |\phi_k\rangle,$$

that is, the system can be divided into k uncorrelated parts.

Examples for four qubits:

$$
\begin{aligned}
|\psi_{fs}\rangle &= |0000\rangle && \text{is fully separable,}\\
|\psi_{ts}\rangle &= |00\rangle \otimes (|00\rangle + |11\rangle) && \text{is 3-separable,}\\
|\psi_{bs}\rangle &= |0\rangle \otimes (|000\rangle + |111\rangle) && \text{is biseparable,}\\
|GHZ_4\rangle &= |0000\rangle + |1111\rangle && \text{is truly multipartite entangled.}
\end{aligned}
$$

A. Acin, D. Bruß, M. Lewenstein, A. Sanpera, PRL 87, 040401 (2001).

Classifications

Problem

Task: Simplify an N-particle state via local operations,

$$|\psi\rangle \mapsto T_A \otimes T_B \otimes \cdots \otimes T_N |\psi\rangle$$

- The number of parameters of all T_i scales linearly.
- The number of parameters of $|\psi\rangle$ scales exponentially.

$\Rightarrow$ There will be a continuum of equivalence classes.

Classifications

Problem

Task: Simplify an N-particle state via local operations,

$$|\psi\rangle \mapsto T_A \otimes T_B \otimes \cdots \otimes T_N |\psi\rangle$$

- The number of parameters of all T_i scales linearly.
- The number of parameters of $|\psi\rangle$ scales exponentially.

$\Rightarrow$ There will be a continuum of equivalence classes.

Result for qubits

- Make the single-qubit marginals diagonal in z basis.
- In the generic case $\varrho_I \neq \mathbb{1}/2$: $\Rightarrow$ Only one candidate for LU
- If $\varrho_I = \mathbb{1}/2$: One can also also decide LU equivalence

B. Kraus, PRL 104, 020504 (2010)

Maximally entangled states

How entangled can two couples get?

A. Higuchi, A. Sudbery *

Dept. of Mathematics, University of York, Heslington, York, YO10 5DD, UK

How entangled can two couples get?

A. Higuchi, A. Sudbery *

Dept. of Mathematics, University of York, Heslington, York, YO10 5DD, UK

Results and Questions

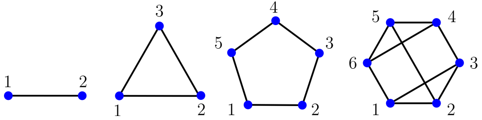
- A bipartite pure state is maximally entangled, if the marginals are maximally mixed.
- For four qubits, there is no state that is maximally entangled for any bipartition.
- What happens for general states of N particles?

Phys. Lett. A 273, 213 (2000)

Absolutely Maximally Entangled states

Results on AME states

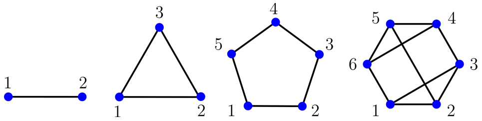
- An N-particle state where all $\lfloor N/2 \rfloor$-particle reduced states are maximally mixed is called AME.
- Examples: Bell states, GHZ states, quantum codewords, ...



Absolutely Maximally Entangled states

Results on AME states

- An N-particle state where all $\lfloor N/2 \rfloor$-particle reduced states are maximally mixed is called AME.
- Examples: Bell states, GHZ states, quantum codewords, ...



- AME states correspond to $((N, 1, \lfloor N/2 \rfloor + 1))_D$ quantum codes.
- If D is large enough, they exist for any N.
- Qubits: They exist for $N = 2, 3, 5, 6$ but not for $N = 4$ and $N \geq 8$.
- So what happens for $N = 7$?

The seven qubit case

First result

There is no AME state for seven qubits.

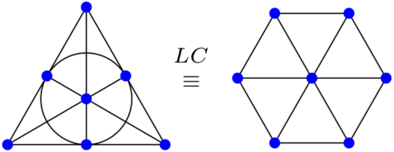
The seven qubit case

First result

There is no AME state for seven qubits.

Second result

The best approximation to a seven qubit AME state is a graph state where 32 of the 35 three-body density matrices are maximally mixed.



F. Huber et al., PRL 118, 200502 (2017).

AME(4,6)

Another long-standing question

Does there exist an AME of four six-dimensional systems?

This is a quantum version of Euler's problem of orthogonal lattice squares.

AME(4,6)

Another long-standing question

Does there exist an AME of four six-dimensional systems?

This is a quantum version of Euler's problem of orthogonal lattice squares.

Final result

This AME state exists!

S. A. Rather, PRL 128, 080507 (2022); D. Garisto, Quanta Magazine 2022

General strategies

Rains' shadow inequality

Consider positive operators X and Y on N particles and $T \subset \{1, \ldots, N\}$. Then:

$$\sum_{S \subset \{1,\ldots,N\}} (-1)^{|S \cap T|} Tr_S \left[Tr_{S^c}(X) Tr_{S^c}(Y) \right] \geq 0$$

General strategies

Rains' shadow inequality

Consider positive operators X and Y on N particles and $T \subset \{1, \ldots, N\}$. Then:

$$\sum_{S \subset \{1,\ldots,N\}} (-1)^{|S \cap T|} Tr_S \left[Tr_{S^c}(X) Tr_{S^c}(Y) \right] \geq 0$$

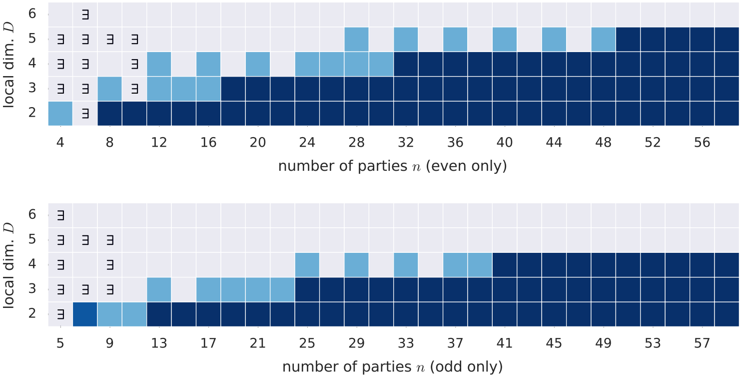
Application to the AME problem

- Assume that an AME state $|\psi\rangle$ exists and set $X = Y = |\psi\rangle\langle\psi|$.
- Since $|\psi\rangle$ is AME, many $\left[Tr_{S^c}(X)^2 \right]$ in the SI are known as proportional to the identity.
- If one finds a contradiction, the AME does not exist.

E.M. Rains, IEEE Trans. Inf. Theory 46, 54 (2000); F. Huber et al., JPA 51, 175301 (2018)

General results

Using similar ideas and the theory of weight and shadow enumerators we can exclude many more cases:



F. Huber et al., JPA 51, 175301 (2018), for updates see: `http://www.tp.nt.uni-siegen.de/+fhuber/ame.html`.

Entanglement measures

The three tangle

The three tangle

Question

A state $|\psi\rangle$ in the W class is characterized by the possibility that

$$|W\rangle = A \otimes B \otimes C|\psi\rangle$$

Which algebraic constraints for $|\psi\rangle$ guarantee a solution?

The three tangle

Question

A state $|\psi\rangle$ in the W class is characterized by the possibility that

$$|W\rangle = A \otimes B \otimes C|\psi\rangle$$

Which algebraic constraints for $|\psi\rangle$ guarantee a solution?

Three tangle

For $|\psi\rangle = \sum_{ijk} a_{ijk}|ijk\rangle$ we have:

$$\tau_3(|\psi\rangle) = 4|d_1 - 2d_2 + 4d_3|$$

$$d_1 = a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2;$$

$$\begin{aligned} d_2 = {} & a_{000} a_{111} a_{011} a_{100} + a_{000} a_{111} a_{101} a_{010} \\ & + a_{000} a_{111} a_{110} a_{001} + a_{011} a_{100} a_{101} a_{010} \\ & + a_{011} a_{100} a_{110} a_{001} + a_{101} a_{010} a_{110} a_{001}; \end{aligned}$$

$$d_3 = a_{000} a_{110} a_{101} a_{011} + a_{111} a_{001} a_{010} a_{100}.$$

This is nonzero iff $|\psi\rangle$ is in the GHZ class.

V. Coffman et al., PRA 61, 052306 (2000)

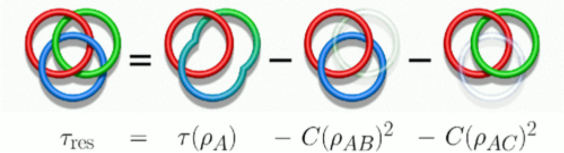
Monogamy

For any three-qubit state, one has

$$C^2_{A|BC}(|\psi\rangle) = C^2_{A|B}(\varrho_{AB}) + C^2_{A|C}(\varrho_{AC}) + \tau_3(|\psi\rangle)$$

with C^2 being the squared concurrence.

- For GHZ: $C^2_{A|B} = C^2_{A|C} = 0$, so τ_3 is maximal.
- For W: $C^2_{A|B}$ and $C^2_{A|C}$ large and $\tau_3 = 0$.



V. Coffman et al., PRA 61, 052306 (2000), T. J. Osborne, F. Verstraete, PRL 96, 220503 (2006); C. Eltschka, J. Siewert, PRL 114, 140402 (2015)

The geometric measure

The geometric measure

Definition

Given a pure multipartite state $|\varphi\rangle$, define

$$\Lambda^2(\varphi) = \max_{|a\rangle,|b\rangle,|c\rangle} |\langle a, b, c|\varphi\rangle|^2$$

Then the geometric measure is given by

$$E_G(\varphi) = 1 - \Lambda^2$$

The geometric measure

Definition

Given a pure multipartite state $|\varphi\rangle$, define

$$\Lambda^2(\varphi) = \max_{|a\rangle,|b\rangle,|c\rangle} |\langle a, b, c|\varphi\rangle|^2$$

Then the geometric measure is given by

$$E_G(\varphi) = 1 - \Lambda^2$$

Remarks

- For mixed states $\varrho = \sum_k p_k|\varphi_k\rangle\langle\varphi_k|$, one takes the convex roof

$$E_G(\varrho) = \min_{p_k,\varphi_k} \sum_k p_k E_G(\varphi_k)$$

- For simplicity, we will focus in this talk on two and three parties.

T.C. Wei, P.M. Goldbart, PRA 68, 042307 (2003); L. Weinbrenner et al., arXiv:2505.01394.

Examples

- For a bipartite state $|\varphi\rangle = \sum_i s_i |ii\rangle$ we have

$$\Lambda^2(\varphi) = s_1^2 = \text{squared maximal Schmidt coefficient}$$

Examples

- For a bipartite state $|\varphi\rangle = \sum_i s_i |ii\rangle$ we have

$$\Lambda^2(\varphi) = s_1^2 = \text{squared maximal Schmidt coefficient}$$

- For the GHZ state,

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

  one gets an upper bound $\Lambda^2 \leq 1/2$ by considering bipartitions $|\psi_{ab}\rangle|c\rangle$ and the corresponding Schmidt coefficients.

- This bound can be reached by the state $|000\rangle$:

$$\Rightarrow \Lambda^2(GHZ) = \frac{1}{2}$$

Examples

- For symmetric states $|\varphi_{\mathrm{sym}}\rangle = \Pi_S|\varphi_{\mathrm{sym}}\rangle$ the closest product state is symmetric

$$\Lambda^2(\varphi_{\mathrm{sym}}) = \max_{|a\rangle} |\langle a, a, a|\varphi_{\mathrm{sym}}\rangle|^2$$

Examples

- For symmetric states $|\varphi_{\mathrm{sym}}\rangle = \Pi_S|\varphi_{\mathrm{sym}}\rangle$ the closest product state is symmetric

$$\Lambda^2(\varphi_{\mathrm{sym}}) = \max_{|a\rangle} |\langle a, a, a|\varphi_{\mathrm{sym}}\rangle|^2$$

- For the W state,

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

this leads to a one-parameter optimization, resulting in

$$\Rightarrow \Lambda^2(W) = \frac{4}{9} < \frac{1}{2}$$

The W state is more entangled than the GHZ state.

R. Hübener et al., PRA 80, 032324 (2009)

Numerical computation

Simple iteration

- Take $|\psi\rangle$ and fix $|b\rangle$ for Bob and $|c\rangle$ for Charlie.
- Compute for Alice the state

$$|\chi\rangle = \langle b|\langle c|\psi\rangle$$

Numerical computation

Simple iteration

- Take $|\psi\rangle$ and fix $|b\rangle$ for Bob and $|c\rangle$ for Charlie.
- Compute for Alice the state

$$|\chi\rangle = \langle b|\langle c|\psi\rangle$$

- The optimal $|a\rangle$ is proportional to $|\chi\rangle$

$$|a\rangle = \frac{1}{\mathcal{N}}|\chi\rangle$$

- Then: Fix $|a\rangle$ and $|c\rangle$ and compute the optimal $|b\rangle$. Iterate!
- This works well for up to eight qubits.

O. Gühne et al., PRL 98, 110502 (2007); S. Gerke et al., PRX 8, 031047 (2018)

Properties

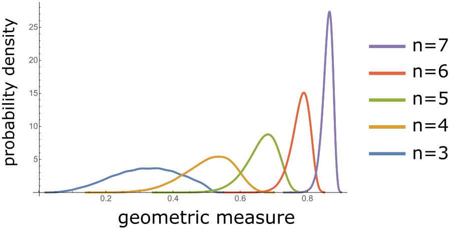
- Interpretation: A set of states $\{|\psi_i\rangle\}$ with high geometric measure is difficult to distinguish locally.

  M. Hayashi et al., PRL 96, 040501 (2006).

- The quantity $\Lambda^2(\psi)$ is also called the injective tensor norm, related to tensor eigenvalues

  A. Montanaro, "Injective tensor norms and open problems in QI"; L. Qi, J. Symb. Comput. 40, 1302 (2005)

- Scaling of Λ^2: Generic states of many particles are highly entangled

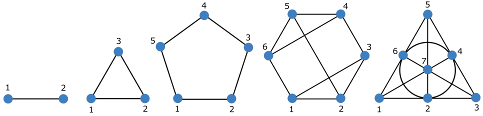


M. J. Bremner et al., PRL 102, 190502 (2009); D. Gross et al., PRL 102, 190501 (2009).

What are maximally entangled states?

n	G_{max}	$\lvert\varphi\rangle_{\text{max}}$
2	$1/2$	$\lvert\psi^-\rangle$
3	$0.5555 \approx 5/9$	$\lvert\text{W}\rangle$
4	$0.7777 \approx 7/9$	$\lvert\text{M}\rangle$
5	$0.8686 \approx (1/36)(33 - \sqrt{3})$	$\lvert G_5\rangle$
6	$0.9166 \approx 11/12$	$\lvert G_6\rangle$
7	≥ 0.941	$\text{MMS}(7,2)$

$$|\text{M}\rangle = \frac{1}{\sqrt{3}}\left(|GHZ_4\rangle + e^{2\pi i/3}\sigma_x^{(3)}\sigma_x^{(4)}|GHZ_4\rangle + e^{4\pi i/3}\sigma_x^{(2)}\sigma_x^{(4)}|GHZ_4\rangle\right)$$



J. Steinberg et al., PRA 110, 062428 (2024)

Possible generalizations

Instead of fully product states one can also consider the overlap

$$\Theta^2(\varphi) = \max_{|\eta\rangle} |\langle\eta|\varphi\rangle|^2$$

with

- biseparable states,

$$|\eta\rangle = \{|\psi_{ab}\rangle|c\rangle \text{ or } |a\rangle|\psi_{bc}\rangle \text{ or } |\psi_{ac}\rangle|b\rangle\}$$

- states with fixed tensor rank

$$|\eta\rangle = |a, b, c\rangle + |\alpha, \beta, \gamma\rangle$$

- superpositions of biseparable states

$$|\eta\rangle = |\psi_{ab}\rangle|c\rangle + |a\rangle|\psi_{bc}\rangle + |\psi_{ac}\rangle|b\rangle$$

These and other problems (like LU optimization) are mathematically more or less equivalent to the original problem.

S. Denker, I. Septembre, in preparation.

Conclusion

- There is no clear "maximally entangled state" for more than two particles.
- There are different forms of multiparticle entanglement.
- Multiparticle entanglement is closely related to open problems in mathematics.

Literature

- O. Gühne, G. Toth, Entanglement detection, Physics Reports 474, 1 (2009), arXiv:0811.2803.
- L. Weinbrenner, O. Gühne, Quantifying entanglement from the geometric perspective, arXiv:2505.01394.