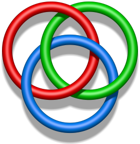
Multiparticle Quantum Entanglement II

Otfried Gühne

Department Physik, Universität Siegen

|TQO⟩ Theoretical Quantum Optics



General Structure

1. Lecture I: Pure states
2. Lecture II: Mixed states
3. Lecture III: Graph states and other families of states

Schedule for Lecture II

1. Entanglement measures
2. Recap: Bipartite mixed states
3. Multiparticle Entanglement of mixed states
4. Network entanglement

Entanglement measures

The three tangle

The three tangle

Question

A state $|\psi\rangle$ in the W class is characterized by the possibility that

$$|W\rangle = A \otimes B \otimes C|\psi\rangle$$

Which algebraic constraints for $|\psi\rangle$ guarantee a solution?

The three tangle

Question

A state $|\psi\rangle$ in the W class is characterized by the possibility that

$$|W\rangle = A \otimes B \otimes C|\psi\rangle$$

Which algebraic constraints for $|\psi\rangle$ guarantee a solution?

Three tangle

For $|\psi\rangle = \sum_{ijk} a_{ijk}|ijk\rangle$ we have:

$$\tau_3(|\psi\rangle) = 4|d_1 - 2d_2 + 4d_3|$$

$$d_1 = a_{000}^2 a_{111}^2 + a_{001}^2 a_{110}^2 + a_{010}^2 a_{101}^2 + a_{100}^2 a_{011}^2;$$

$$\begin{aligned} d_2 = {} & a_{000}a_{111}a_{011}a_{100} + a_{000}a_{111}a_{101}a_{010} \\ & + a_{000}a_{111}a_{110}a_{001} + a_{011}a_{100}a_{101}a_{010} \\ & + a_{011}a_{100}a_{110}a_{001} + a_{101}a_{010}a_{110}a_{001}; \end{aligned}$$

$$d_3 = a_{000}a_{110}a_{101}a_{011} + a_{111}a_{001}a_{010}a_{100}.$$

This is nonzero iff $|\psi\rangle$ is in the GHZ class.

V. Coffman et al., PRA 61, 052306 (2000)

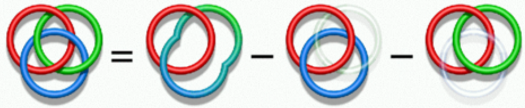
Monogamy

For any three-qubit state, one has

$$C^2_{A|BC}(|\psi\rangle) = C^2_{A|B}(\varrho_{AB}) + C^2_{A|C}(\varrho_{AC}) + \tau_3(|\psi\rangle)$$

with C^2 being the squared concurrence.

- For GHZ: $C^2_{A|B} = C^2_{A|C} = 0$, so τ_3 is maximal.
- For W: $C^2_{A|B}$ and $C^2_{A|C}$ large and $\tau_3 = 0$.



$$\tau_{\rm res} = \tau(\rho_A) - C(\rho_{AB})^2 - C(\rho_{AC})^2$$

V. Coffman et al., PRA 61, 052306 (2000), T. J. Osborne et al., PRL 96, 220503 (2006); C. Eltschka et al., PRL 114, 140402 (2015)

The geometric measure

The geometric measure

Definition

Given a pure multipartite state $|\varphi\rangle$, define

$$\Lambda^2(\varphi) = \max_{|a\rangle,|b\rangle,|c\rangle} |\langle a, b, c|\varphi\rangle|^2$$

Then the geometric measure is given by

$$E_G(\varphi) = 1 - \Lambda^2$$

The geometric measure

Definition

Given a pure multipartite state $|\varphi\rangle$, define

$$\Lambda^2(\varphi) = \max_{|a\rangle,|b\rangle,|c\rangle} |\langle a, b, c|\varphi\rangle|^2$$

Then the geometric measure is given by

$$E_G(\varphi) = 1 - \Lambda^2$$

Remarks

- For mixed states $\varrho = \sum_k p_k |\varphi_k\rangle\langle\varphi_k|$, one takes the convex roof

$$E_G(\varrho) = \min_{p_k,\varphi_k} \sum_k p_k E_G(\varphi_k)$$

- For simplicity, I focus on two and three parties.

T.C. Wei, P.M. Goldbart, PRA 68, 042307 (2003); L. Weinbrenner et al., arXiv:2505.01394.

Examples

- For a bipartite state $|\varphi\rangle = \sum_i s_i |ii\rangle$ we have

$$\Lambda^2(\varphi) = s_1^2 = \text{squared maximal Schmidt coefficient}$$

Examples

- For a bipartite state $|\varphi\rangle = \sum_i s_i |ii\rangle$ we have

$$\Lambda^2(\varphi) = s_1^2 = \text{squared maximal Schmidt coefficient}$$

- For the GHZ state,

$$|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$$

  one gets an upper bound $\Lambda^2 \leq 1/2$ by considering bipartitions $|\psi_{ab}\rangle|c\rangle$ and the corresponding Schmidt coefficients.

- This bound can be reached by the state $|000\rangle$:

$$\Rightarrow \Lambda^2(GHZ) = \frac{1}{2}$$

Examples

- For symmetric states $|\varphi_{\rm sym}\rangle = \Pi_S|\varphi_{\rm sym}\rangle$ the closest product state is symmetric
$$\Lambda^2(\varphi_{\rm sym}) = \max_{|a\rangle} |\langle a, a, a|\varphi_{\rm sym}\rangle|^2$$

Examples

- For symmetric states $|\varphi_{\rm sym}\rangle = \Pi_S|\varphi_{\rm sym}\rangle$ the closest product state is symmetric

$$\Lambda^2(\varphi_{\rm sym}) = \max_{|a\rangle} |\langle a, a, a|\varphi_{\rm sym}\rangle|^2$$

- For the W state,

$$|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$$

this leads to a one-parameter optimization, resulting in

$$\Rightarrow \Lambda^2(W) = \frac{4}{9} < \frac{1}{2}$$

The W state is more entangled than the GHZ state.

R. Hübener et al., PRA 80, 032324 (2009)

Numerical computation

Simple iteration

- Take $|\psi\rangle$ and fix $|b\rangle$ for Bob and $|c\rangle$ for Charlie.
- Compute for Alice the state

$$|\chi\rangle = \langle b|\langle c|\psi\rangle$$

Numerical computation

Simple iteration

- Take $|\psi\rangle$ and fix $|b\rangle$ for Bob and $|c\rangle$ for Charlie.
- Compute for Alice the state

$$|\chi\rangle = \langle b|\langle c|\psi\rangle$$

- The optimal $|a\rangle$ is proportional to $|\chi\rangle$

$$|a\rangle = \frac{1}{\mathcal{N}}|\chi\rangle$$

- Then: Fix $|a\rangle$ and $|c\rangle$ and compute the optimal $|b\rangle$. Iterate!
- This works well for up to eight qubits.

O. Gühne et al., PRL 98, 110502 (2007); S. Gerke et al., PRX 8, 031047 (2018)

Properties

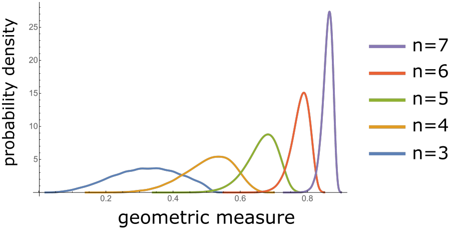
- Interpretation: A set of states $\{|\psi_i\rangle\}$ with high geometric measure is difficult to distinguish locally.

  M. Hayashi et al., PRL 96, 040501 (2006).

- The quantity $\Lambda^2(\psi)$ is also called the injective tensor norm, related to tensor eigenvalues

  A. Montanaro, "Injective tensor norms and open problems in QI"; L. Qi, J. Symb. Comput. 40, 1302 (2005)

- Scaling of Λ^2: Generic states of many particles are highly entangled

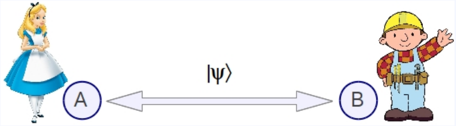


M. J. Bremner et al., PRL 102, 190502 (2009); D. Gross et al., PRL 102, 190501 (2009).

Mixed states of two particles

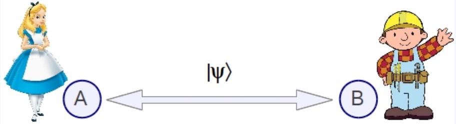
Mathematical formulation

Alice and Bob share a state $|\psi\rangle$.



Mathematical formulation

Alice and Bob share a state $|\psi\rangle$.



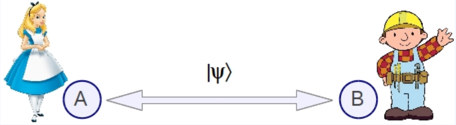
A pure state $|\psi\rangle$ is separable if it is a product state:

$$|\psi\rangle = |a\rangle_A |b\rangle_B = |a, b\rangle.$$

Otherwise it is entangled.

Mathematical formulation

Alice and Bob share a state $|\psi\rangle$.



A pure state $|\psi\rangle$ is separable if it is a product state:

$$|\psi\rangle = |a\rangle_A |b\rangle_B = |a, b\rangle.$$

Otherwise it is entangled.

Mixed states: Consider convex combinations. ϱ is separable if

$$\varrho = \sum_i p_i |a_i\rangle\langle a_i| \otimes |b_i\rangle\langle b_i|, \quad \text{with} \quad p_i \geq 0, \ \sum_i p_i = 1.$$

Interpretation: Entanglement cannot be generated by local operations and classical communication.

R. Werner, PRA 40, 4277 (1989).

The separability problem

Open question

Given ϱ, is it entangled or separable?

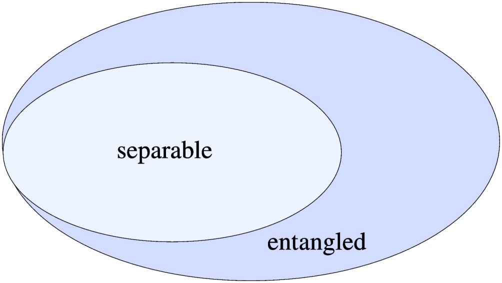
The separability problem

Open question

Given ϱ, is it entangled or separable?

Geometrical interpretation

The set of all separable states is convex.



The PPT criterion

Are there simple criteria to prove that a state is entangled?

The PPT criterion

Are there simple criteria to prove that a state is entangled?

Transposition and partial transposition

- Transposition: The usual transposition $X \mapsto X^T$ does not change the eigenvalues of the matrix X
- For a product space one can also consider the partial transposition. If $X = A \otimes B$:

$$X^{T_B} = A \otimes B^T$$

The PPT criterion

Are there simple criteria to prove that a state is entangled?

Transposition and partial transposition

- Transposition: The usual transposition $X \mapsto X^T$ does not change the eigenvalues of the matrix X
- For a product space one can also consider the partial transposition. If $X = A \otimes B$:

$$X^{T_B} = A \otimes B^T$$

Partial transposition and separability

Theorem. If a state is separable, then its partial transposition has no negative eigenvalues ("the state is PPT" or $\varrho^{T_B} \geq 0$).
Proof:

$$\varrho_{sep}^{T_B} = \sum_k p_k \varrho_A \otimes \varrho_B^T = \sum_k p_k \varrho_A \otimes \tilde{\varrho}_B \geq 0.$$

The PPT criterion

Are there simple criteria to prove that a state is entangled?

Transposition and partial transposition

- Transposition: The usual transposition $X \mapsto X^T$ does not change the eigenvalues of the matrix X
- For a product space one can also consider the partial transposition. If $X = A \otimes B$:
$$X^{T_B} = A \otimes B^T$$

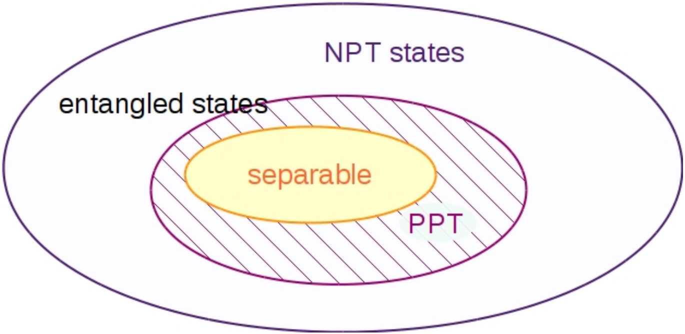
Partial transposition and separability

Theorem. If a state is separable, then its partial transposition has no negative eigenvalues ("the state is PPT" or $\varrho^{T_B} \geq 0$).
Proof:
$$\varrho_{sep}^{T_B} = \sum_k p_k \varrho_A \otimes \varrho_B^T = \sum_k p_k \varrho_A \otimes \tilde{\varrho}_B \geq 0.$$

Remark: For two qubits: ϱ is PPT $\Leftrightarrow$ ϱ is separable.

A. Peres, PRL 77, 1413 (1996), Horodecki$^{\otimes 3}$, PLA 223, 1(1996)

Geometry



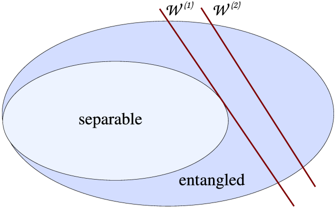
Entanglement witnesses

An observable $\mathcal{W}$ is an entanglement witness, if

$$Tr(\mathcal{W}\varrho)\begin{cases} \geq 0 & \text{for all separable } \varrho_s, \\ < 0 & \text{for one entangled } \varrho_e. \end{cases}$$

If $Tr(\mathcal{W}\varrho)$ is measured:

$$Tr(\mathcal{W}\varrho)\begin{cases} < 0 \Rightarrow & \varrho \text{ is entangled}, \\ \geq 0 \Rightarrow & \text{no detection}. \end{cases}$$



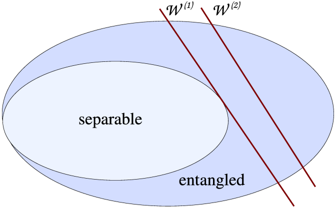
Entanglement witnesses

An observable $\mathcal{W}$ is an entanglement witness, if

$$Tr(\mathcal{W}\varrho) \begin{cases} \geq 0 & \text{for all separable } \varrho_s, \\ < 0 & \text{for one entangled } \varrho_e. \end{cases}$$

If $Tr(\mathcal{W}\varrho)$ is measured:

$$Tr(\mathcal{W}\varrho) \begin{cases} < 0 \Rightarrow & \varrho \text{ is entangled}, \\ \geq 0 \Rightarrow & \text{no detection}. \end{cases}$$



- For any entangled ϱ there is a witness.
- Witnesses can be optimized ($\mathcal{W}^{(1)}$ optimal, $\mathcal{W}^{(2)}$ not!).
- Witnesses assume correct measurements, contrary to Bell inequalities

A simple example

- Consider a Bell state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

- A witness is given by

$$\mathcal{W} = \frac{\mathbb{1}}{2} - |\psi^-\rangle\langle\psi^-|.$$

- Interpretation: If the fidelity $F = Tr(|\psi^-\rangle\langle\psi^-|\varrho)$ exceeds $F = 1/2$:
  $\Rightarrow \varrho$ is entangled!

M. Bourennane et al., PRL 92, 087902 (2004).

A simple example

- Consider a Bell state

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle).$$

- A witness is given by

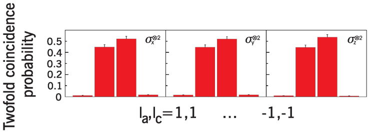
$$\mathcal{W} = \frac{\mathbb{1}}{2} - |\psi^-\rangle\langle\psi^-|.$$

- Interpretation: If the fidelity $F = Tr(|\psi^-\rangle\langle\psi^-|\varrho)$ exceeds $F = 1/2$:
  $\Rightarrow$ ϱ is entangled!

- Local decomposition:

$$\mathcal{W} = \frac{1}{4}(\mathbb{1}\otimes\mathbb{1} + \sigma_x\otimes\sigma_x + \sigma_y\otimes\sigma_y + \sigma_z\otimes\sigma_z).$$

- Measurement gives:



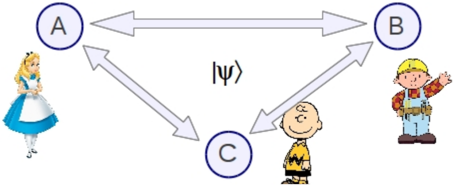
$$Tr(\mathcal{W}\varrho) = -0.461 \pm 0.003.$$

The state is entangled and $F = 0.941$!

M. Bourennane et al., PRL 92, 087902 (2004).

Multipartite entanglement of mixed states

Multiparticle entanglement



There are different possibilities:

- Fully separable:

$$|\psi^{\mathrm{fs}}\rangle = |000\rangle$$

- Biseparable:

$$|\psi^{\mathrm{bs}}\rangle = |0\rangle \otimes (|00\rangle + |11\rangle)$$

- Genuine multiparticle entangled:

$$|GHZ\rangle = |000\rangle + |111\rangle \quad \text{or} \quad |W\rangle = |001\rangle + |010\rangle + |100\rangle.$$

- Mixed states: Convex combinations, again.

Examples and details

- Fully separable state:

$$\varrho_{\rm fs} = \sum_k p_k |a_k b_k c_k\rangle\langle a_k b_k c_k|.$$

- Biseparable state:

$$\varrho_{\rm bs} = \frac{1}{4}|\psi^-_{AB}\rangle\langle\psi^-_{AB}| \otimes |0_C\rangle\langle 0_C| + \frac{3}{4}|1_A\rangle\langle 1_A| \otimes |\phi^+_{BC}\rangle\langle\phi^+_{BC}|.$$

- W class state:

$$\varrho_{\rm W} = \frac{8}{9}|W\rangle\langle W| + \frac{1}{9}|010\rangle\langle 010|.$$

Examples and details

- Fully separable state:

$$\varrho_{\rm fs} = \sum\nolimits_k p_k |a_k b_k c_k\rangle\langle a_k b_k c_k|.$$

- Biseparable state:

$$\varrho_{\rm bs} = \frac{1}{4}|\psi^-_{AB}\rangle\langle\psi^-_{AB}| \otimes |0_C\rangle\langle 0_C| + \frac{3}{4}|1_A\rangle\langle 1_A| \otimes |\phi^+_{BC}\rangle\langle\phi^+_{BC}|.$$

- W class state:

$$\varrho_{\rm W} = \frac{8}{9}|W\rangle\langle W| + \frac{1}{9}|010\rangle\langle 010|.$$

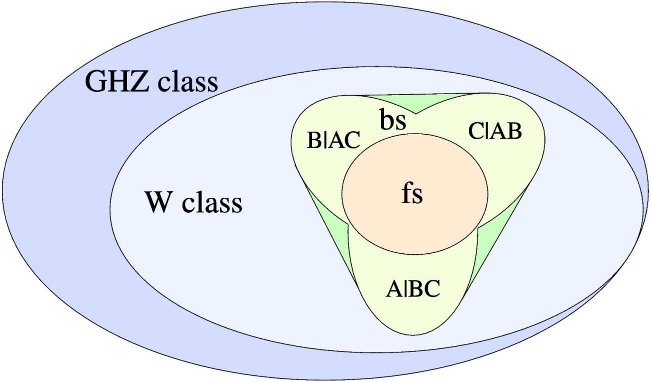
- The GHZ state mixed with white noise,

$$\varrho(p) = p|GHZ\rangle\langle GHZ| + (1-p)\frac{\mathbb{1}}{8}$$

is fully separable for $p \leq 1/5$, biseparable for $p \leq 3/7$, in the W class for $p < 0.6955$ and in the GHZ class elsewhere.

C. Eltschka at al., PRL 108, 020502 (2012)

Classification of mixed three-qubit states

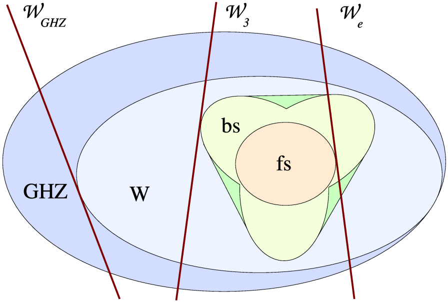


- The class of mixed W states is not of measure zero.
- States can be separable for any bipartition, but not fully separable.

A. Acin, D. Bruß, M. Lewenstein, A. Sanpera, PRL 87, 040401 (2001).

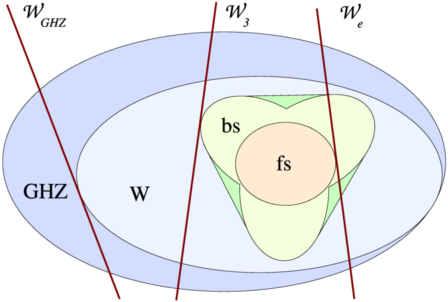
Witnesses for multiparticle entanglement

Witnesses for different classes of entanglement:



Witnesses for multiparticle entanglement

Witnesses for different classes of entanglement:



A typical witness for $|\psi\rangle$ is

$$\mathcal{W} = \alpha \mathbb{1} - |\psi\rangle\langle\psi|.$$

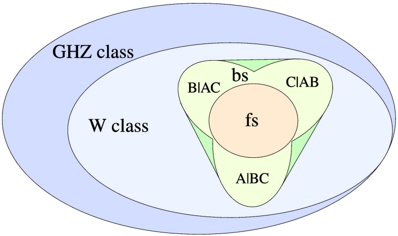
e.g., $\mathcal{W} = \mathbb{1}/2 - |GHZ\rangle\langle GHZ|$.

Problem

Separability criteria

- There are simple criteria for two particles (e.g. PPT)
- Can they be generalized to more particles?
- The problem are mixtured of different bipartitions:

$$\varrho^{\mathrm{bs}} = p_1 \varrho^{\mathrm{sep}}_{A|BC} + p_2 \varrho^{\mathrm{sep}}_{B|AC} + p_3 \varrho^{\mathrm{sep}}_{C|AB}.$$



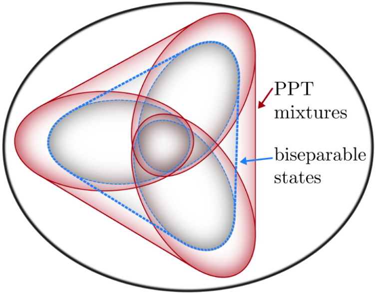
Idea

Replace separable states by PPT states. Instead of biseparable states,

$$\varrho^{\mathrm{bs}} = p_1 \varrho_{A|BC}^{\mathrm{sep}} + p_2 \varrho_{B|AC}^{\mathrm{sep}} + p_3 \varrho_{C|AB}^{\mathrm{sep}},$$

consider PPT mixtures:

$$\varrho^{\mathrm{pmix}} = p_1 \varrho_{A|BC}^{\mathrm{ppt}} + p_2 \varrho_{B|AC}^{\mathrm{ppt}} + p_3 \varrho_{C|AB}^{\mathrm{ppt}}.$$



- This is an SDP
- Often necessary and sufficient
- This can quantify multipartite entanglement

B. Jungnitsch et al. PRL 2011, M. Hofmann et al. JPA 2014

Semidefinite programming

Semidefinite programming

$$\text{minimize: } \vec{c}^T \vec{x}$$

$$\text{subject to: } F_0 + \sum_i x_i F_i \geq 0$$

Here: $\vec{x}$ are variables, $\vec{c}$ coefficients, F_i matrices.

- For SDPs, certified solutions can be found (duality).
- In practice, one can solve them with few lines of code (Mosek).

Semidefinite programming

$$\text{minimize: } \vec{c}^T \vec{x}$$

$$\text{subject to: } F_0 + \sum_i x_i F_i \geq 0$$

Here: $\vec{x}$ are variables, $\vec{c}$ coefficients, F_i matrices.

- For SDPs, certified solutions can be found (duality).
- In practice, one can solve them with few lines of code (Mosek).

SDP in QIT

- Given an observable W, what is the maximal $Tr(\varrho W)$ for ϱ PPT?

Semidefinite programming

$$\text{minimize: } \vec{c}^T \vec{x}$$

$$\text{subject to: } F_0 + \sum_i x_i F_i \geq 0$$

Here: $\vec{x}$ are variables, $\vec{c}$ coefficients, F_i matrices.

- For SDPs, certified solutions can be found (duality).
- In practice, one can solve them with few lines of code (Mosek).

SDP in QIT

- Given an observable W, what is the maximal $Tr(\varrho W)$ for ϱ PPT?
- Given marginals ϱ_{AB} and ϱ_{BC}, is there a global ϱ_{ABC} with these marginals?

Semidefinite programming

$$\text{minimize: } \vec{c}^T \vec{x}$$

$$\text{subject to: } F_0 + \sum_i x_i F_i \geq 0$$

Here: $\vec{x}$ are variables, $\vec{c}$ coefficients, F_i matrices.

- For SDPs, certified solutions can be found (duality).
- In practice, one can solve them with few lines of code (Mosek).

SDP in QIT

- Given an observable W, what is the maximal $Tr(\varrho W)$ for ϱ PPT?
- Given marginals ϱ_{AB} and ϱ_{BC}, is there a global ϱ_{ABC} with these marginals? Both SDPs!
- Given marginals ϱ_{AB} and ϱ_{BC}, is there a pure $|\psi\rangle_{ABC}$ with these marginals?

Semidefinite programming

$$\text{minimize: } \vec{c}^T \vec{x}$$

$$\text{subject to: } F_0 + \sum_i x_i F_i \geq 0$$

Here: $\vec{x}$ are variables, $\vec{c}$ coefficients, F_i matrices.

- For SDPs, certified solutions can be found (duality).
- In practice, one can solve them with few lines of code (Mosek).

SDP in QIT

- Given an observable W, what is the maximal $Tr(\varrho W)$ for ϱ PPT?
- Given marginals ϱ_{AB} and ϱ_{BC}, is there a global ϱ_{ABC} with these marginals? Both SDPs!
- Given marginals ϱ_{AB} and ϱ_{BC}, is there a pure $|\psi\rangle_{ABC}$ with these marginals? No SDP!

The resulting method

Classification via witnesses

A state ϱ is not a PPT mixture, if and only if $Tr(\varrho\mathcal{W}) < 0$ for

$$\mathcal{W} = P_A + Q_A^{T_A} = P_B + Q_B^{T_B} = P_C + Q_C^{T_C}$$

with $P_i, Q_i \geq 0$.

The resulting method

Classification via witnesses

A state ϱ is not a PPT mixture, if and only if $Tr(\varrho\mathcal{W}) < 0$ for

$$\mathcal{W} = P_A + Q_A^{T_A} = P_B + Q_B^{T_B} = P_C + Q_C^{T_C}$$

with $P_i, Q_i \geq 0$.

Main advantages

- This can be solved via semidefinite programming.
- In practice, it requires only few lines of code in Mathematica
- Numerically, it works for ≤ 7 qubits. Analytically, up to "∞" qubits.
- The amount of the violation is an entanglement monotone.

Results

Noise robustness

The noise robustness increases drastically: Consider

$$\varrho(p) = p\mathbb{1}/8 + (1-p)|\psi\rangle\langle\psi|$$

and compute maximal p_{tol} :

state	tolerances p_{tol}	
	new	before
$\|GHZ_3\rangle^\star$	0.571	0.571
$\|GHZ_4\rangle^\star$	0.533	0.533
$\|W_3\rangle^\star$	0.521	0.421
$\|W_4\rangle$	0.526	0.444
$\|Cl_4\rangle^\star$	0.615	0.533
$\|D_{2,4}\rangle$	0.539	0.381
$\|\Psi_{S,4}\rangle$	0.553	0.317

B. Jungnitsch et al., PRL 106, 190502 (2011).

Results

Noise robustness

The noise robustness increases drastically: Consider

$$\varrho(p) = p\mathbb{1}/8 + (1-p)|\psi\rangle\langle\psi|$$

and compute maximal p_{tol} :

state	tolerances p_{tol}	
	new	before
$\lvert GHZ_3\rangle^\star$	0.571	0.571
$\lvert GHZ_4\rangle^\star$	0.533	0.533
$\lvert W_3\rangle^\star$	0.521	0.421
$\lvert W_4\rangle$	0.526	0.444
$\lvert Cl_4\rangle^\star$	0.615	0.533
$\lvert D_{2,4}\rangle$	0.539	0.381
$\lvert \Psi_{S,4}\rangle$	0.553	0.317

B. Jungnitsch et al., PRL 106, 190502 (2011).

Permutation invariant states

For PI states of three qubits

$$\varrho = \pi_{ij}\varrho\pi_{ij}$$

the PPT mixer is necessary and sufficient for entanglement.

L.Novo et al., PRA 2013

Extensions

Similar criteria for multiparticle entanglement based on other bipartite criteria, such as the computable cross norm / realignment criterion.

C. Zhang, S. Denker et al., PRL 2024

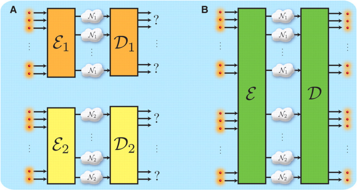
Superactivation

Superactivation

Two broken phones are useless

Superactivation

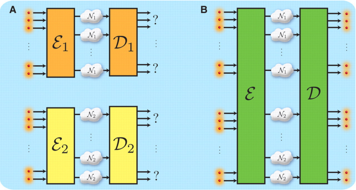
Two broken quantum channels may be useful!



G. Smith and J. Yard, Science 2008

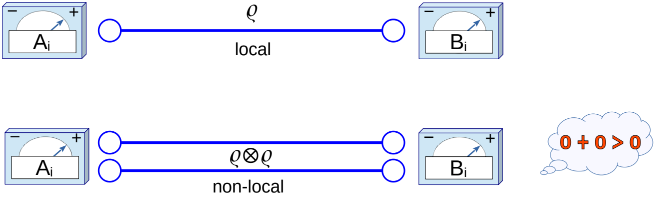
Superactivation

Two broken quantum channels may be useful!



G. Smith and J. Yard, Science 2008

Two local quantum states can be nonlocal!



C. Palazuelos, PRL 2012

Superactivation of GME

Tensor stability

Does a property of ϱ hold for the two-copy state $\varrho^{\otimes 2}$, too?

- Bipartite separability: ϱ_{AB} separable $\Rightarrow$ $\varrho_{AB}^{\otimes 2}$ separable
- PPT: ϱ_{AB} is PPT $\Rightarrow$ $\varrho_{AB}^{\otimes 2}$ is PPT.
- But: TS criteria don't detect much entanglement in high dim.

There are PPT entangled states far away from the separable states, S. Beigi, P. W. Shor, JMP 51, 042202 2010.

Superactivation of GME

Tensor stability

Does a property of ϱ hold for the two-copy state $\varrho^{\otimes 2}$, too?

- Bipartite separability: ϱ_{AB} separable $\Rightarrow$ $\varrho_{AB}^{\otimes 2}$ separable
- PPT: ϱ_{AB} is PPT $\Rightarrow$ $\varrho_{AB}^{\otimes 2}$ is PPT.
- But: TS criteria don't detect much entanglement in high dim.

There are PPT entangled states far away from the separable states, S. Beigi, P. W. Shor, JMP 51, 042202 2010.

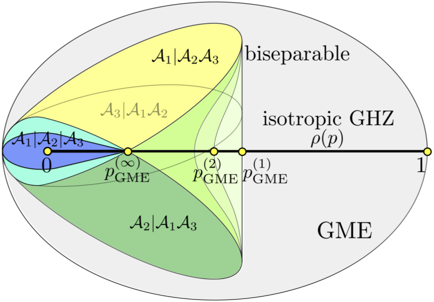
Is biseparability tensor stable?

For a biseparable state

$$\varrho = p_1 \varrho_{AB} \otimes \varrho_C + p_2 \sigma_A \otimes \sigma_{BC} + p_3 \tau_B \otimes \tau_{AB}$$

the two-copy state $\varrho^{\otimes 2}$ contains cross terms and is not clearly biseparable.

Superactivation of GME



- GME can be superactivated.
- *Any* state that is not separable for a fixed partition becomes GME for many copies.
- What does this mean for multiparticle entanglement as a resource?

H. Yamasaki et al., Quantum 6, 695 (2022); C. Palazuelos et al., Quantum 6, 735 (2022); L. Weinbrenner et al., arXiv:2412.18331.

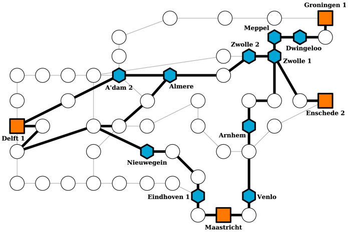
Network Entanglement

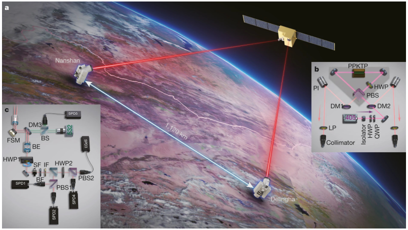
Quantum networks

Many people dream of global quantum communication

Quantum networks

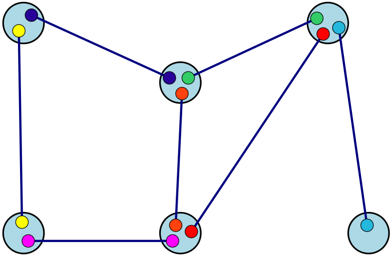
Many people dream of global quantum communication





J. Rabbie et al., Nature QI 2022; J. Yin et al., Nature 2020.

Theorist's perspective

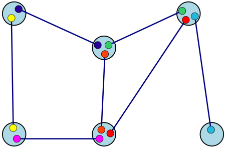


- Network of quantum nodes with physical links.
- Entanglement is created along the links with some imperfections.
- Which types of quantum correlations arise in this network?
- Networks also provide a paradigm to study quantum nonlocality.

C. Branciard et al., PRA 2012, N. Gisin et al., Nature Comm. 2020, A. Tavakoli et al., arXiv:2104.10700

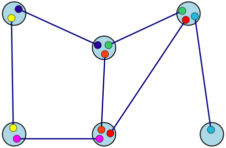
Basic idea

Consider a multipartite scenario. If a state can be generated by distributing two-particle source states only, then it is not multiparticle entangled.



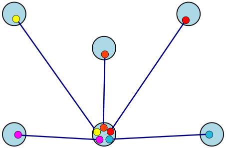
Basic idea

Consider a multipartite scenario. If a state can be generated by distributing two-particle source states only, then it is not multiparticle entangled.



Problem

If LOCC are allowed, then any state can be prepared via teleportation.

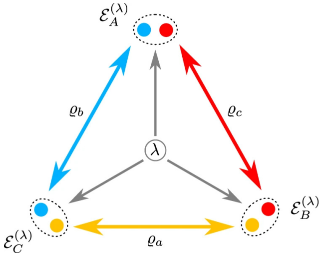


⇒ One has to restrict the available local operations.

Network entanglement

LOSR paradigm

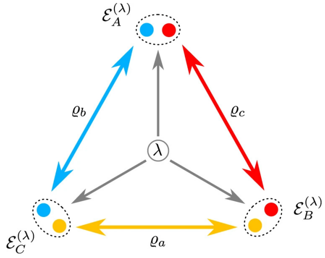
Can a state be prepared using local operations & shared randomness?



Network entanglement

LOSR paradigm

Can a state be prepared using local operations & shared randomness?

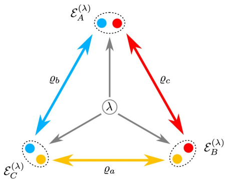


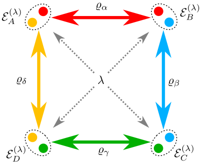
Formally: Can the quantum state be written as:

$$\varrho \stackrel{?}{=} \sum_{\lambda} p_\lambda \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)} [\varrho_a \otimes \varrho_b \otimes \varrho_c]$$

M. Navascues et al, PRL 2020; M. X. Luo, Adv. Quantum Tech. 2021; T. Kraft et al, PRA 2021

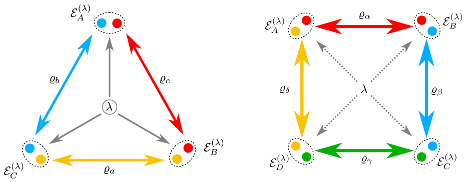
Network entanglement





$$\varrho \stackrel{?}{=} \sum_\lambda p_\lambda \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)}[\varrho_a \otimes \varrho_b \otimes \varrho_c]$$

Network entanglement



$$\varrho \stackrel{?}{=} \sum_\lambda p_\lambda \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)}[\varrho_a \otimes \varrho_b \otimes \varrho_c]$$

Remarks

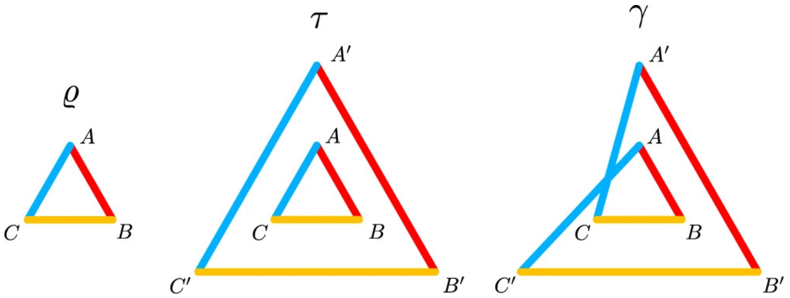
- The source states ϱ_x may be high-dimensional.
- Randomness λ can be shifted from the maps to the source states.
- No communication allowed (or possible)
- This procedure can generate genuine multipartite entanglement.

K. Hansenne et al., Nature Comm. 2022

Quantum inflation

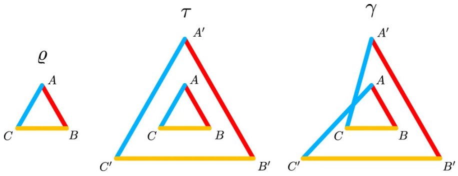
Idea

If a state can be generated in a network, one can consider multiple copies of the sources, which may be wired differently.



E. Wolfe et al., PRX 2021; M. Navascues et al., PRL 2020, L. Ligthart et al., CMP 2023

Quantum inflation



Properties

- The inflations share some marginals, e.g.,

$$\tau_{ABC} = \tau_{A'B'C'} = \varrho, \quad \gamma_{A'C} = \tau_{AC} = \varrho_{AC}, \quad \gamma_{AC} = \tau_{A'C}$$

- The search for γ and τ with such properties is an SDP, can be tackled analytically or numerically.
- We obtain fidelity bounds

$$F_{GHZ} \leq 0.618, \quad F_{CL} \leq 0.7377.$$

M. Navascues et al, PRL 2020, K. Hansenne et al., Nat. Comm. 2022

Generalizations

Observations

- These methods are difficult to extend to many particles.
- One would expect: If large quantum states are considered, fidelity bounds go exponentially down.

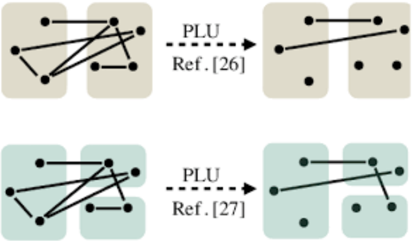
Generalizations

Observations

- These methods are difficult to extend to many particles.
- One would expect: If large quantum states are considered, fidelity bounds go exponentially down.

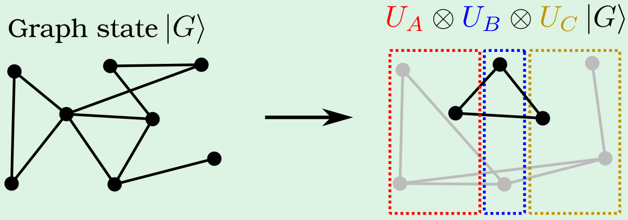
Idea

If a multi-qubit state can be prepared, this may imply that GHZ states can be prepared in triangle scenarios.



Results

- No graph state can be prepared better than the GHZ in the triangle.



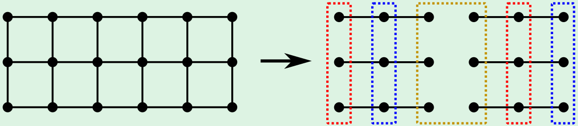
$$\Rightarrow F_{\mathrm{LOSR}}(|\mathrm{G}\rangle) < 0.62$$

J. Neumann et al., arXiv:2503.09473

Results

- Cluster states and other families are exponentially hard to prepare.

Cluster state $|\mathrm{Cl}_{m,n}\rangle$

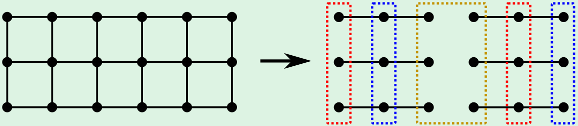


$$\Rightarrow F_{\mathrm{LOSR}}(|\mathrm{Cl}_{m,n}\rangle) \lesssim 0.9^{mn}$$

Results

- Cluster states and other families are exponentially hard to prepare.

Cluster state $|\mathrm{Cl}_{m,n}\rangle$



$$\Rightarrow F_{\mathrm{LOSR}}(|\mathrm{Cl}_{m,n}\rangle) \lesssim 0.9^{mn}$$

Main message

- There is a fundamental difference between distributed bipartite entanglement and multiparticle entanglement.
- Communication and quantum memories are essential for networks.

J. Neumann et al., arXiv:2503.09473, see also X. Zhou et al., arXiv:2503.09480

Conclusion

- There are different and inequivalent measures of multiparticle entanglement.
- GME can be characterized by generalizations of the PPT criterion.
- GME can be superactivated.
- Networks pose interesting problems for characterizing correlations.

Literature

- L. Weinbrenner et al., arXiv:2505.01394.
- B. Jungnitsch et al., Phys. Rev. Lett. 106, 190502 (2011), arXiv:1010.6049.
- K. Hansenne, Z.P. Xu et al., Nature Comm. 13, 496 (2022), arXiv:2108.02732.