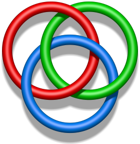
Multiparticle Quantum Entanglement III

Otfried Gühne



Department Physik, Universität Siegen

General Structure

1. Lecture I: Pure states
2. Lecture II: Mixed states
3. Lecture III: Graph states and other families of states

Schedule for Lecture III

1. Graph states
2. Network entanglement
3. Hypergraph states

Graph states

What are the interesting multiqubit states?

- The GHZ states violate Bell inequalities maximally:

$$|GHZ\rangle = |0000\rangle + |1111\rangle$$

- The W-states are robust against qubit loss:

$$|W\rangle = |1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle$$

- The cluster states are useful for the one-way quantum computer:

$$|CL\rangle = |0000\rangle + |1100\rangle + |0011\rangle - |1111\rangle$$

- The Dicke states are often easy to prepare:

$$|D\rangle = |0011\rangle + |0101\rangle + |1001\rangle + |0110\rangle + |1010\rangle + |1100\rangle$$

- The singlet states are $U \otimes ... \otimes U$ invariant:

$$|\psi^{(4)}\rangle = |0011\rangle + |1100\rangle - \frac{1}{2}(|10\rangle + |10\rangle) \otimes (|10\rangle + |10\rangle)$$

Motivation

The GHZ state $|GHZ\rangle = |000\rangle + |111\rangle$ is an eigenstate of

$$\mathbb{1}\mathbb{1}\mathbb{1} \,|\, ZZ\mathbb{1},\ Z\mathbb{1}Z,\ XXX \,|\, \mathbb{1}ZZ,\ -YYX,\ -YXY,\ -XYY$$

Can this be generalized?

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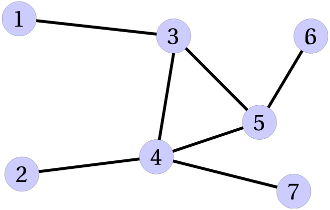
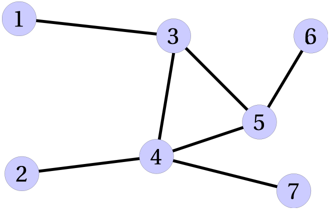
Facts

- These are $2^3 = 8$ commuting observables, forming a group.
- Group is generated by $Z\mathbb{1}Z, ZZ\mathbb{1}, XXX$, also by YYX, YXY, XYY.
- These observables were useful for deriving Bell inequalities.

Graph states as stabilizer states



Graph states as stabilizer states

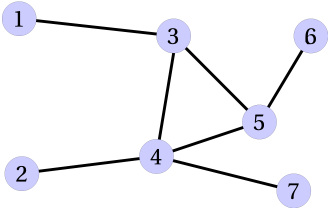


- For any graph, define stabilizing operators as

$$g_i = X_i \bigotimes_{j \in N(i)} Z_j.$$

These commute, have eigenvalues ± 1 and a common eigenbasis.

Graph states as stabilizer states



- For any graph, define stabilizing operators as

$$g_i = X_i \bigotimes_{j \in N(i)} Z_j.$$

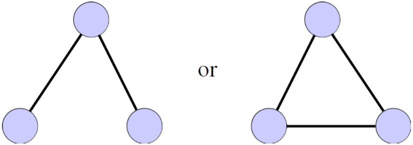
These commute, have eigenvalues ± 1 and a common eigenbasis.

- The graph state $|G\rangle$ is the unique state fulfilling

$$g_i |G\rangle = |G\rangle.$$

M. Hein, J. Eisert, H.J. Briegel, PRA 69, 062311 (2004)

Example



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$$XZ\mathbb{1},\ ZXZ,\ \mathbb{1}ZX \quad \text{or} \quad XZZ,\ ZXZ,\ ZZX$$

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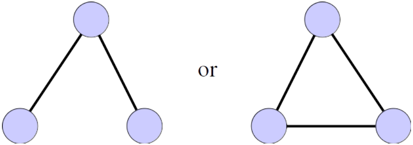

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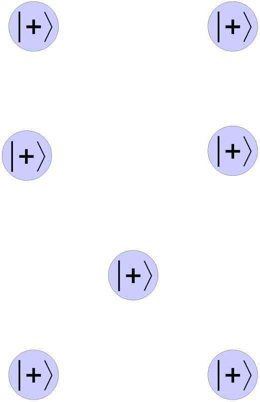
$$XZ\mathbb{1},\ ZXZ,\ \mathbb{1}ZX \quad \text{or} \quad XZZ,\ ZXZ,\ ZZX$$

- Up to some relabeling, these are the generating sets from above.
- $\Rightarrow$ The GHZ state is a graph state, with two potential graphs.

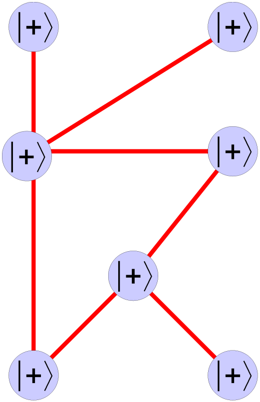
Graph states



(1) Start with N qubits in the state

$$|+\rangle = (|0\rangle + |1\rangle)$$

Graph states



(1) Start with N qubits in the state

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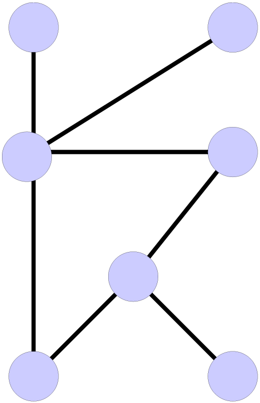
(2) Apply on the edges a phase gate:

$$C_e = \mathbb{1} - 2|11\rangle\langle 11|$$

This is an Ising-type interaction.

Graph states



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(3) Resulting state is the graph state.

M. Hein, J. Eisert, H.J. Briegel, PRA 69, 062311 (2004).

Graph states

Technical points

- Since phase gates commute, one may also write

$$|G\rangle = \prod_{e \in E} C_e |+\rangle^{\otimes N}$$

- To see the equivalence of the definitions, note that

$$X_1 Z_2 = C_{\{1,2\}} X_1 C_{\{1,2\}}$$

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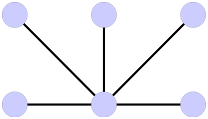
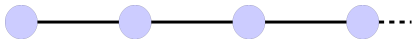
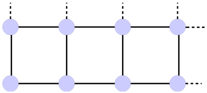
Further examples

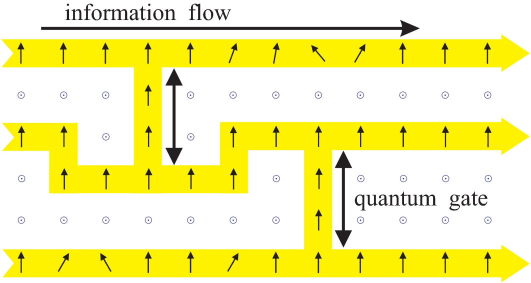
General GHZ states, 1D and 2D cluster states,







Application: Measurement-based quantum computation



- By making local measurement on a cluster state, a quantum computer can be realized.
- Problem: Experimental generation of the cluster state.

R. Raussendorf, H. Briegel, PRL 86, 5188 (2001).

Further applications

- All code words in quantum error correcting codes correspond to graph states.

  D. Schlingemann and R.F. Werner, PRA 65, 012308 (2002).

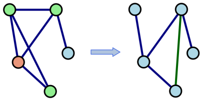
- GHZ-type arguments and Bell inequalities can be derived for arbitrary graph states.

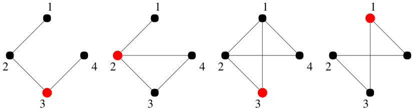
  O. Gühne et al., PRL 95, 120405 (2005).

- Other potential applications: Secret sharing, multiparty quantum cryptography, quantum metrology, ...

Local equivalences: LU, LC

- Local Clifford unitaries map Pauli matrices to Pauli matrices.
- Action of LC can be described by local complementation:





M. van den Nest et al., PRA 69, 022316 (2004).

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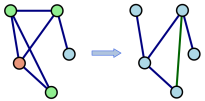

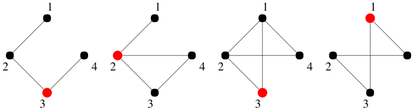

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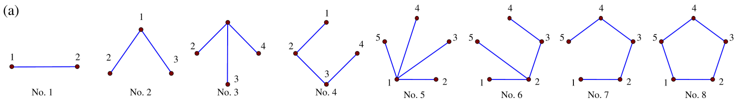
- This leads to a classification of $N \leq 11$ qubits.



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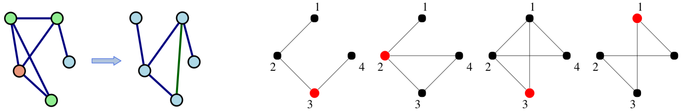

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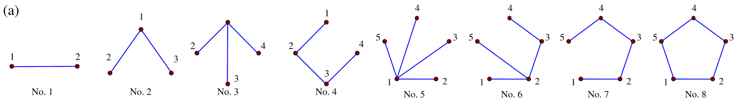

- But: For $N = 27$ there are LU equivalent graph states that are not LC equivalent.

Z. Ji et al., QIC 10, 97 (2010), N. Tsimakuridze, O. Gühne, J. Phys. A 2017

Stabilizer

- The products of the g_i form the commutative stabilizer

$$S(G) = \{s_j, j = 1, ..., 2^N\}; \quad s_j = \prod_{i \in I_j} g_i$$

D. Gottesman, Phys. Rev. A 54, 1862 (1996).

- *Any* commutative subgroup of the Pauli group with 2^N elements can be represented by a graph (up to local Clifford).

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- Considering the other eigenvalues gives the graph state basis,

$$g_i |G_{\vec{r}}\rangle = (-1)^{r_i} |G_{\vec{r}}\rangle.$$

- A very useful formula:

$$|G\rangle\langle G| = \prod_{i=1}^{N} \frac{\mathbb{1} + g_i}{2} = \frac{1}{2^N} \sum_{j=1}^{2^N} s_j.$$

Depolarization

Given a state ϱ, what is

$$\sigma = \frac{1}{2^N} \sum_i s_i \varrho s_i ?$$

This can also obtained as

$$\varrho \mapsto \tau_1 = \frac{1}{2}(\varrho + g_1 \varrho g_1) \mapsto \tau_2 = \frac{1}{2}(\tau_1 + g_2 \tau g_2) \mapsto \cdots \mapsto \sigma.$$

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Properties

- This is a sequence of LOCC, entanglement decays.
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Properties

- This is a sequence of LOCC, entanglement decays.
- If σ is entangled, then ϱ was also entangled.
- If we write $\varrho = \sum_{ij} \alpha_{ij} |G_i\rangle\langle G_j|$ in the graph-state basis

$$\varrho \mapsto \sigma = \sum_i \alpha_i |G_i\rangle\langle G_i|$$

- $\Rightarrow$ Graph-diagonal states are interesting!

GHZ-diagonal states

For three-qubits, one can consider the GHZ basis,

$$|000\rangle \pm |111\rangle, \quad |001\rangle \pm |110\rangle, \quad |010\rangle \pm |101\rangle, \quad |100\rangle \pm |011\rangle$$

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The GHZ-diagonal states are of X-form:

$$\varrho = \begin{bmatrix} \lambda_1 & & & \cdots & \cdots & & & \mu_1 \\ & \lambda_2 & & & & & \mu_2 & \\ & & \lambda_3 & & & \mu_3 & & \\ & & & \lambda_4 & \mu_4 & & & \\ & & & \mu_4 & \lambda_4 & & & \\ & & \mu_3 & & & \lambda_3 & & \\ & \mu_2 & & & & & \lambda_2 & \\ \mu_1 & & & \cdots & \cdots & & & \lambda_1 \end{bmatrix}$$

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For these states, many things have been solved.
Depolarization to GHZ-diagonal states is useful also for experimental data.

W. Dür et al., JPA 2001, O. Gühne et al., NJP 2010; Z.H. Ma et al., PRA 2011; S. M. Hashemi Rafsanjani et al., PRA 2012; C. Eltschka et al, PRL 2012

Network Entanglement

Quantum networks

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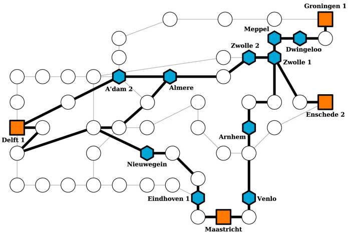
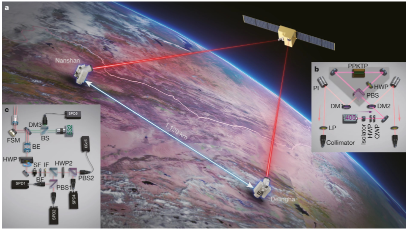
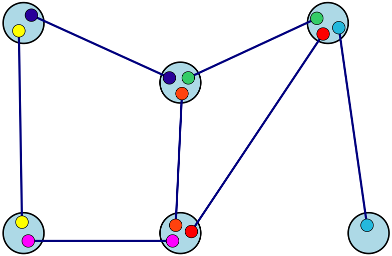




J. Rabbie et al., Nature QI 2022; J. Yin et al., Nature 2020.

Theorist's perspective

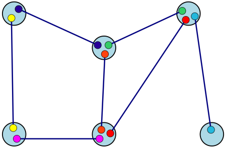


- Network of quantum nodes with physical links.
- Entanglement is created along the links with some imperfections.
- Which types of quantum correlations arise in this network?
- Networks also provide a paradigm to study quantum nonlocality.

C. Branciard et al., PRA 2012, N. Gisin et al., Nature Comm. 2020, A. Tavakoli et al., arXiv:2104.10700

Basic idea

Consider a multipartite scenario. If a state can be generated by distributing two-particle source states only, then it is not multiparticle entangled.



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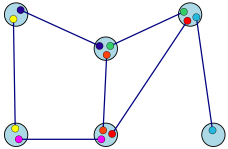
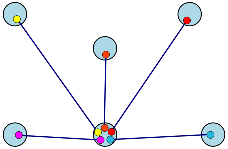


Problem

If LOCC are allowed, then any state can be prepared via teleportation.

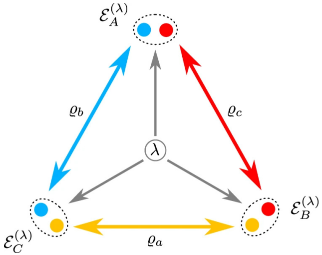


$\Rightarrow$ One has to restrict the available local operations.

Network entanglement

LOSR paradigm

Can a state be prepared using local operations & shared randomness?



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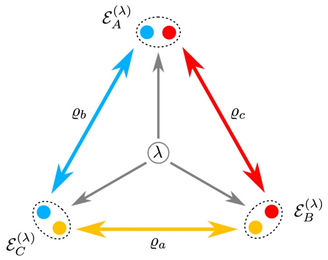
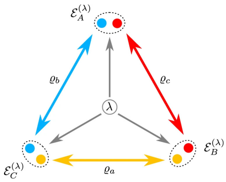


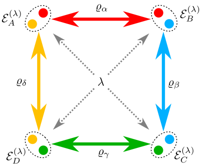
Formally: Can the quantum state be written as:

$$\varrho \stackrel{?}{=} \sum_{\lambda} p_{\lambda} \mathcal{E}_A^{(\lambda)} \otimes \mathcal{E}_B^{(\lambda)} \otimes \mathcal{E}_C^{(\lambda)} [\varrho_a \otimes \varrho_b \otimes \varrho_c]$$

M. Navascues et al, PRL 2020; M. X. Luo, Adv. Quantum Tech. 2021; T. Kraft et al, PRA 2021

Network entanglement





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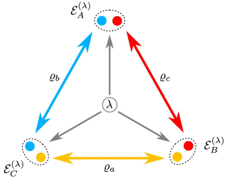
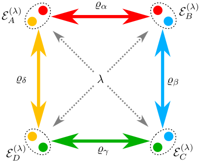
Network entanglement





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Remarks

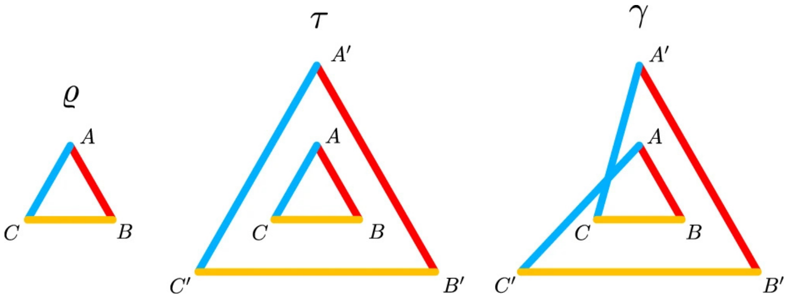
- The source states ϱ_x may be high-dimensional.
- Randomness λ can be shifted from the maps to the source states.
- No communication allowed (or possible)
- This procedure can generate genuine multipartite entanglement.

K. Hansenne et al., Nature Comm. 2022

Quantum inflation

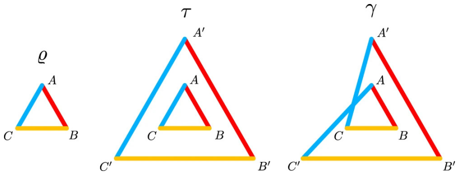
Idea

If a state can be generated in a network, one can consider multiple copies of the sources, which may be wired differently.



E. Wolfe et al., PRX 2021; M. Navascues et al., PRL 2020, L. Ligthart et al., CMP 2023

Quantum inflation



Properties

- The inflations share some marginals, e.g.,

$$\tau_{ABC} = \tau_{A'B'C'} = \varrho, \quad \gamma_{A'C} = \tau_{AC} = \varrho_{AC}, \quad \gamma_{AC} = \tau_{A'C}$$

- The search for γ and τ with such properties is an SDP, can be tackled analytically or numerically.
- We obtain fidelity bounds

$$F_{GHZ} \leq 0.618, \quad F_{CL} \leq 0.7377.$$

M. Navascues et al, PRL 2020, K. Hansenne et al., Nat. Comm. 2022

Generalizations

Observations

- These methods are difficult to extend to many particles.
- One would expect: If large quantum states are considered, fidelity bounds go exponentially down.

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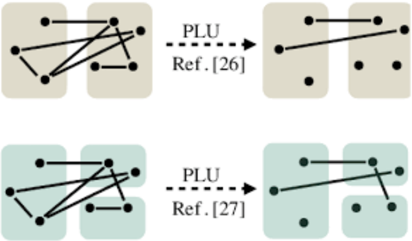
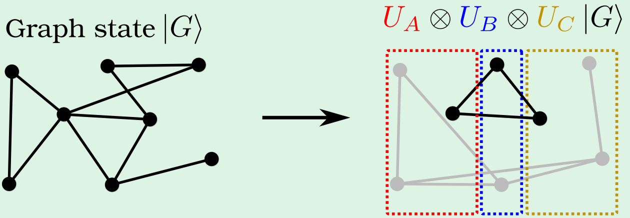
Idea

If a multi-qubit state can be prepared, this may imply that GHZ states can be prepared in triangle scenarios.



Results

- No graph state can be prepared better than the GHZ in the triangle.



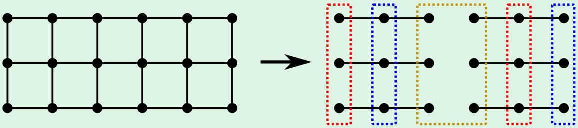
$$\Rightarrow F_{\mathrm{LOSR}}(|\mathrm{G}\rangle) < 0.62$$

J. Neumann et al., arXiv:2503.09473

Results

- Cluster states and other families are exponentially hard to prepare.

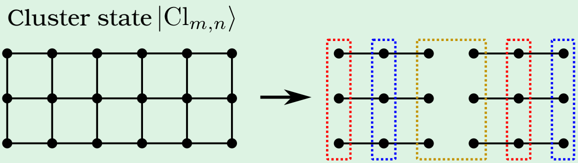
Cluster state $|\mathrm{Cl}_{m,n}\rangle$



$$\Rightarrow F_{\mathrm{LOSR}}(|\mathrm{Cl}_{m,n}\rangle) \lesssim 0.9^{mn}$$

Results

- Cluster states and other families are exponentially hard to prepare.



$$\Rightarrow F_{\mathrm{LOSR}}(|\mathrm{Cl}_{m,n}\rangle) \lesssim 0.9^{mn}$$

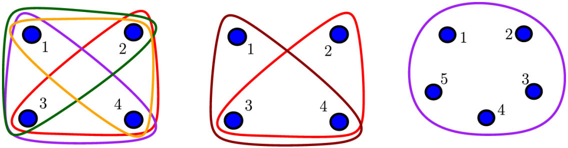
Main message

- There is a fundamental difference between distributed bipartite entanglement and multiparticle entanglement.
- Communication and quantum memories are essential for networks.

J. Neumann et al., arXiv:2503.09473, see also X. Zhou et al., arXiv:2503.09480

Hypergraph states

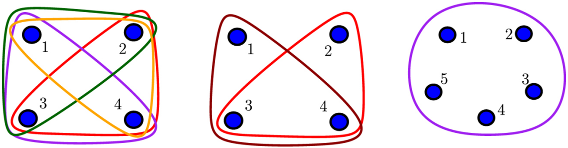
Basic definitions



(a) (b) (c)

In a hypergraph, edges can contain more than two vertices.

Basic definitions



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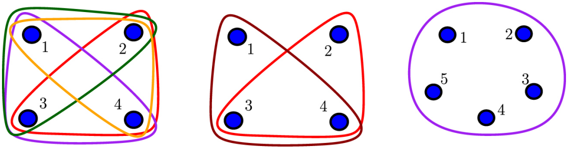

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The hypergraph state is:

$$|H\rangle = \prod_{e\in E} C_e |+\rangle^{\otimes N}$$

C. Kruszynska, B. Kraus. PRA 79, 052304 (2009), M. Rossi, M. Huber, D. Bruß,C. Macchiavello, NJP 15 (2013).

The nonlocal stabilizer

Define for each qubit the operator

$$g_i \equiv \Big(\prod_{e\in E} C_e\Big) X_i \Big(\prod_{e\in E} C_e\Big) = X_i \otimes \Big(\prod_{e\ni i} C_{e\setminus\{i\}}\Big)$$

Then:

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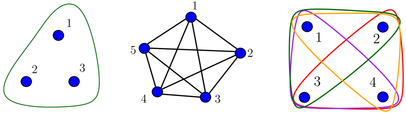
The stabilizing operators g_i:

- ... are hermitean, but nonlocal,
- ... commute: $g_i g_j = g_j g_i$,
- ... generate a group with 2^N elements.

Examples

The three-qubit HG state



For the simplest nontrivial HG we have

$$|H_3\rangle = \frac{1}{\sqrt{8}}(|000\rangle + |001\rangle + |010\rangle + |011\rangle + |100\rangle + |101\rangle + |110\rangle - |111\rangle)$$

after Hadamard transformation on the third qubit:

$$|H_3\rangle = \frac{1}{2}(|000\rangle + |010\rangle + |100\rangle + |111\rangle).$$

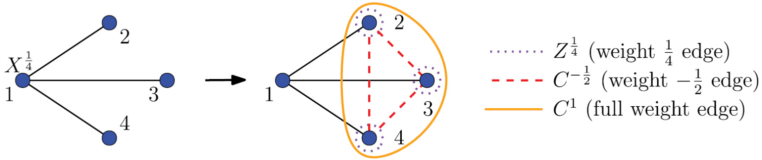
This state was also called "logical AND state".

S. Abramsky, C. Costantin, arXiv:1412.5213

HG states as a tool

Local operations

For HG states, one can derive graphical rules for some transformations



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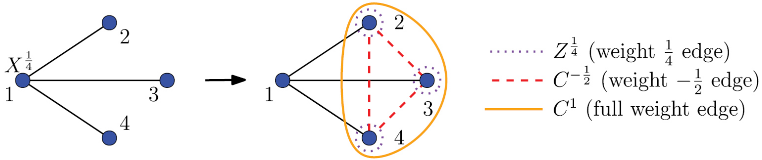
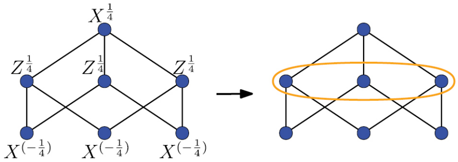


LU equivalence

Using these rules: HG states can be LU equivalent to graph states



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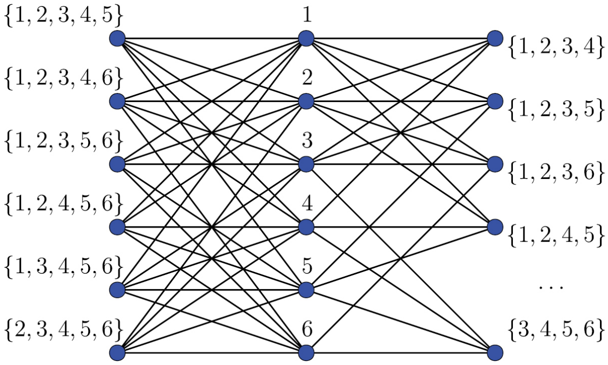
HG states as a tool

LU equivalence

This allows to find counterexamples to the LU-LC conjecture.



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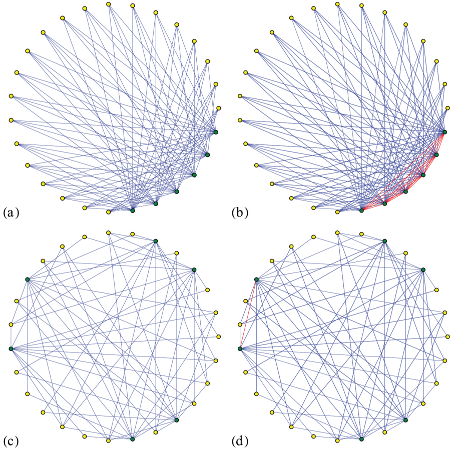

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Bell inequalities for hypergraph states

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First Problem

Can the non-local stabilizer be used for characterizing local correlations?

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- The state $|H_3\rangle$ is a $+1$ eigenstate of

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- So we have

$$P(+--|XZZ) = 0.$$

- Furthermore:

$$P(-++|XZZ) + P(-+-|XZZ) + P(--+|XZZ) = 0,$$

$\Rightarrow$ The non-local stabilizer predicts some local perfect correlations!

Hardy argument

If a LHV model satisfies the conditions from zero correlations from the state $|H_3\rangle$ then it must fulfill

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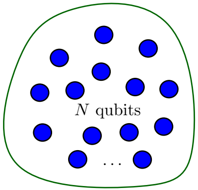
In contrast, for $|H_3\rangle$ we have

$$P(+--|XXX) = \frac{1}{16}$$

This argument can be generalized to N qubits.



Conclusion

- Graph states are a useful family of quantum states with a very elegant description.
- Networks pose interesting problems for characterizing correlations.
- Hypergraph states are a natural extension of graph states

Literature

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- K. Hansenne, Z.P. Xu et al., Nature Comm. 13, 496 (2022); arXiv:2108.02732.
- M Gachechiladze et al., Phys. Rev. Lett. 116, 070401 (2016); arXiv:1507.03570.