Entanglement Lectures: tools and methods

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Anna Sanpera Trigueros (UAB)

UAB
Universitat Autònoma de Barcelona

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Why entanglement?

1. Entanglement is at the heart of quantum physics

2. Quantum information looks for information-processing tasks that (might) offer quantum advantage.

3. Understanding the basic ingredients of quantum advantage forms the foundation for quantum technological applications.

4. ENTANGLEMENT is necessary for many of those applications.

In these lectures:

- First, we will warm up with the postulates of QM & notation
- Define entanglement in (bipartite) pure states
- Look at non- trivial protocols that use pure entangled states.
- Introduce the problem of separability/entanglement for mixed bipartite states.
- Certify entanglement using criteria: partial transposition, majorization, cross-norm, covariance matrix, entanglement witnesses and quantum maps.
- Quantify entanglement.
- Complementary to the lectures of Otfried Ghüne/ Eugene Polzik

Lecture 1

1.1 The Postulates of QM-Recap

POSTULATE 1: Associated to any isolated physical system is a Hilbert space $\mathbb{H}$. The system is completely described by its state vector, $|\psi\rangle \in \mathbb{H}$, which is a unit vector in the system's state space.

POSTULATE 2: The evolution of the state of a closed (isolated) quantum system is given by

$$|\psi(t)\rangle = U(t)|\psi\rangle$$

where $U(t)$ is a unitary operator.

POSTULATE 3: Measurements are described by any collection of operators

$$\left\{ M_m : \mathbb{H} \to \mathbb{H} \mid \sum_m M_m^\dagger M_m = \mathbb{I} \right\}$$

where m denotes the measurement outcome.

If the system is in state $|\psi\rangle \in \mathbb{H}$, then the probability of observing outcome m is given by

$$p_m = \langle\psi|M_m^\dagger M_m|\psi\rangle$$

and the post-measurement state is $|\phi_m\rangle = \dfrac{M_m|\psi\rangle}{\sqrt{p_m}}$

Example: Measuring a qubit: 2 operators M_0 =|0X0| and M_1= |1 X 1|

Postulate 4: The state space of a **composite physical system** is given by the **tensor product** of the state spaces of each of its constituent parts

$$\mathbb{H}_{\text{Total}} = \mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \mathbb{H}_3 \otimes \cdots \otimes \mathbb{H}_N$$

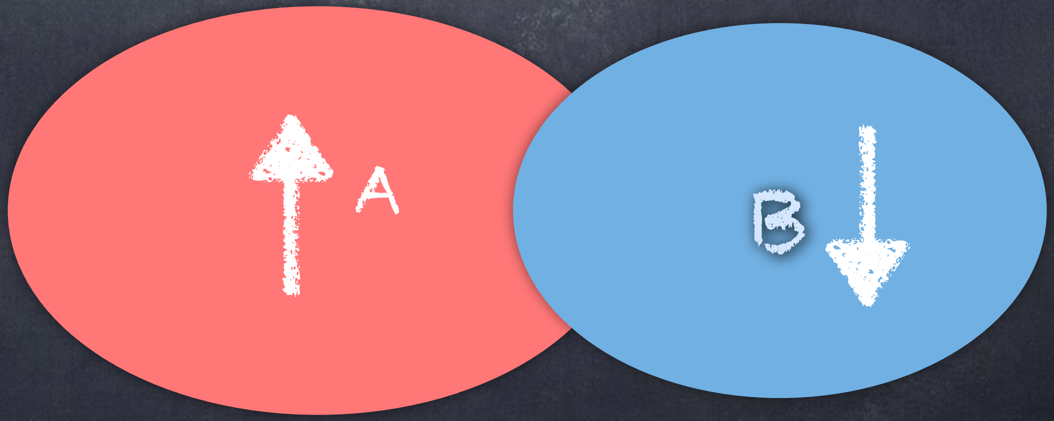
$$\equiv \bigotimes_{k=1}^{N} \mathbb{H}_k$$

Remark: Notice that the tensor product is the ONLY way to preserve the superposition principle and all other q. propertites in composite systems!

System A + B

$\mathbb{H}_{A+B} = \mathbb{H}_A \otimes \mathbb{H}_B$ |01> YES!

$\mathbb{H}_{A+B} \neq \mathbb{H}_A + \mathbb{H}_B$ |0> + |1> NO!



1.2 Composite systems.

Properties of the tensor product

Definition 1: Let $\mathbb{H}_1, \mathbb{H}_2$ be two vector spaces of dimension d_1, d_2 respectively. Suppose that $\{|i_1\rangle\}_{i_1=1}^{d_1}$ is an orthonormal basis of $\mathbb{H}_1$ and $\{|i_2\rangle\}_{i_2=1}^{d_2}$ an orthonormal basis of $\mathbb{H}_2$.

Then an orthonormal basis of $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$ is

$$\{|i_1\rangle \otimes |i_2\rangle\},\ i_1 \in (1, \cdots, d_1),\ i_2 \in (1, \cdots, d_2)$$

Properties of tensor product

1. Let $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$. Then $|\mathbb{H}| = |\mathbb{H}_1| \times |\mathbb{H}_2|$ where $|\mathbb{H}| = d$ denotes the (finite) dimension of the Hilbert space.
2. Whenver the dimension of $\mathbb{H}$ is finite, a Hilbert space is equivalent to a complex vector space. (see lectures of E. Polzik)

3. Suppose $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B$, with $|\phi\rangle_A = \sum_{i_1=1}^{d_1} \phi_{i_1} |i_1\rangle$ and $|\chi\rangle_B = \sum_{i_2=1}^{d_2} \chi_{i_2} |i_2\rangle$

with $\{|i_1\rangle\}_{i_1=1}^{d_1}$, $\{|i_2\rangle\}_{i_2=1}^{d_2}$ orthonormal basis of $\mathbb{H}_1, \mathbb{H}_2$. Then

$$|\psi\rangle = \left(\sum_{i_1=1}^{d_1} \phi_{i_1} |i_1\rangle \right) \otimes \left(\sum_{i_2=1}^{d_2} \chi_{i_2} |i_2\rangle \right) = \sum_{i_1=1}^{d_1} \sum_{i_2=1}^{d_2} \phi_{i_1} \chi_{i_2} |i_1\rangle \otimes |i_2\rangle = \sum_{i_1=1}^{d_1} \sum_{i_2=1}^{d_2} \phi_{i_1} \chi_{i_2} |i_1 i_2\rangle$$

Properties of the tensor product

Equivalently, $|\psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B$, and let $\{|i_1\rangle\}_{i_1=1}^{d_1}$, $\{|i_2\rangle\}_{i_2=1}^{d_2}$ be orthonormal basis of $\mathbb{H}_1$, $\mathbb{H}_2$. Then, the tensorial product of vectors is:

$$|\Psi\rangle_{AB} = \begin{pmatrix} \psi_{11} \\ \psi_{12} \\ \vdots \\ \psi_{1d_2} \\ \psi_{21} \\ \vdots \\ \psi_{2d_2} \\ \vdots \\ \psi_{d_1 1} \\ \vdots \\ \psi_{d_1 d_2} \end{pmatrix} = \begin{pmatrix} \phi_1 \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{d_2} \end{pmatrix} \\ \phi_2 \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{d_2} \end{pmatrix} \\ \vdots \\ \phi_{d_1} \begin{pmatrix} \chi_1 \\ \vdots \\ \chi_{d_2} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} \phi_1\chi_1 \\ \vdots \\ \phi_1\chi_{d_2} \\ \phi_2\chi_1 \\ \vdots \\ \phi_2\chi_{d_2} \\ \vdots \\ \phi_{d_1}\chi_1 \\ \vdots \\ \phi_{d_1}\chi_{d_2} \end{pmatrix}$$

d1 x d2

Properties of the tensor product

1. Let $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$. Then $|\mathbb{H}| = |\mathbb{H}_1| \times |\mathbb{H}_2|$

2. Suppose $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$, and let $\{|i_1\rangle\}_{i_1=1}^{d_1}$, $\{|i_2\rangle\}_{i_2=1}^{d_2}$ be orthonormal basis of $\mathbb{H}_1, \mathbb{H}_2$. Then $|\psi\rangle = \sum_{i_1=1}^{d_1} \sum_{i_2=1}^{d_2} \phi_{i_1} \chi_{i_2} |i_1\rangle \otimes |i_2\rangle$

3. Suppose $B : \mathbb{H}_1 \to \mathbb{H}_1$, $\quad C = \mathbb{H}_2 \to \mathbb{H}_2$. Then $A = B \otimes C$; $\; A : \mathbb{H} \to \mathbb{H}$, is given by

$$A = \left(\sum_{i_1,j_1} B_{i_1,j_1} |i_1\rangle\langle j_1| \right) \otimes \left(\sum_{i_2,j_2} C_{i_2,j_2} |i_2\rangle\langle j_2| \right)$$

$$= \sum_{i_1,j_1} \sum_{i_2,j_2} B_{i_1,j_1} C_{i_2,j_2} \, | i_1, i_2 \rangle \langle j_1, j_2 |$$

Properties of the tensor product

Explicitly, the tensorial product of matrices corresponds to:

$$\begin{pmatrix} A_{11,11} & A_{11,12} & \cdots & A_{11,d_2d_2} \\ A_{12,11} & \cdots & & \\ \vdots & \ddots & & \\ A_{d_1d_1,11} & \cdots & & A_{d_1d_1,d_2d_2} \end{pmatrix} = \begin{pmatrix} B_{11}\begin{pmatrix} C_{11} & \cdots & C_{1d_2} \\ \vdots & \ddots & \\ C_{d_21} & \cdots & C_{d_2d_2} \end{pmatrix} & \cdots & B_{1d_1}\begin{pmatrix} C_{11} & \cdots & C_{1d_2} \\ \vdots & \ddots & \\ C_{d_21} & \cdots & C_{d_2d_2} \end{pmatrix} \\ \vdots & \ddots & \\ B_{d_11}\begin{pmatrix} C_{11} & \cdots & C_{1d_2} \\ \vdots & \ddots & \\ C_{d_21} & \cdots & C_{d_2d_2} \end{pmatrix} & \cdots & B_{d_1d_1}\begin{pmatrix} C_{11} & \cdots & C_{1d_2} \\ \vdots & \ddots & \\ C_{d_21} & \cdots & C_{d_2d_2} \end{pmatrix} \end{pmatrix}$$

Properties of the tensor product

Definition 2: Let $\mathbb{H} = \bigotimes_{i=1}^{N} \mathbb{H}_i$. A unitary $U : \mathbb{H} \to \mathbb{H}$ is said to be a local operation if $U = \otimes_{i=1}^{N} U_i, \quad U_i : \mathbb{H}_i \to \mathbb{H}_i$

Otherwise the operation is said to be non-local unitary.

Definition 3: Let $\mathbb{H} = \bigotimes_{i=1}^{N} \mathbb{H}_i$. A measurement with operators $\left\{M_k : \mathbb{H} \to \mathbb{H}\right\}_{k=1}^{M}$ is said to be local if every measurement operator is of the form

$$M_k = \bigotimes_{i=1}^{N} M_k^{(i)}$$

otherwise the measurement is said to be non-local

Recall: properties of the tensor product

1. Let $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$. Then the dimension of $|\mathbb{H}| = |\mathbb{H}_1| \times |\mathbb{H}_2|$

2. Suppose $|\psi\rangle = |\phi\rangle \otimes |\chi\rangle$, where $|\phi\rangle = \sum_{i_1=1}^{d_1} \phi_{i_1}|i_1\rangle$ and $|\chi\rangle = \sum_{i_2=1}^{d_2} \chi_{i_2}|i_2\rangle$ with $\{|i_1\rangle\}_{i_1=1}^{d_1}$, $\{|i_2\rangle\}_{i_2=1}^{d_2}$ two orthonormal basis of $\mathbb{H}_1, \mathbb{H}_2$. Then

$$|\psi\rangle = \sum_{i_1=1}^{d_1}\sum_{i_2=1}^{d_2} \phi_{i_1}\chi_{i_2}|i_1\rangle \otimes |i_2\rangle$$

3. Suppose $A : \mathbb{H}_1 \to \mathbb{H}_1, \quad B = \mathbb{H}_2 \to \mathbb{H}_2$. Then $C : \mathbb{H} \to \mathbb{H}, C = A \otimes B$ is given by

$$A = \sum_{i_1,j_1}\sum_{i_2,j_2} B_{i_1,j_1}C_{i_2,j_2}|i_1\rangle\langle j_1| \otimes |i_2\rangle\langle j_2| = \sum_{i_1,j_1}\sum_{i_2,j_2} B_{i_1,j_1}C_{i_2,j_2}|i_1 i_2\rangle\langle j_1 j_2|$$

4. The tensor product space $\mathbb{H} = \mathbb{H}_1 \otimes \mathbb{H}_2$ inherits all the properties of its constituent parts (linearity, multiplicative & additive identity etc etc)

Composite systems: meaning

Postulate 4: The state space of a composite physical system is given by the tensor product of the state spaces of each of its constituent parts

$$\mathbb{H}_{\text{Total}} = \mathbb{H}_1 \otimes \mathbb{H}_2 \otimes \mathbb{H}_3 \otimes \cdots \otimes \mathbb{H}_N$$
$$\equiv \bigotimes_{k=1}^{N} \mathbb{H}_k$$

Remark: Tensor products can be used to describe the total state space of a single physical system. E.g., consider describing both the position as well as the angular momentum of a particle

$$\mathbb{H}_{\text{Total}} = \mathbb{H}_{\text{Position}} \otimes \mathbb{H}_{Ang.Mom.}$$

Composite systems

The more prudent way of understanding the tensor product is that it combines together Hilbert spaces associated to distinct properties; (different particles, position, energy, angular momentum of the same particle etc etc etc.)

Remark: We will often omit the tensor symbol entirely writing

$$|i_1\rangle \otimes |i_2\rangle \equiv |i_1 i_2\rangle$$

Definition 4: A composite quantum system is said to be in a product state if

$$|\Psi\rangle = \bigotimes_{i=1}^{N} |\psi_i\rangle$$

where $|\Psi\rangle \in \mathbb{H}_{\text{Total}}$ and $|\psi_i\rangle \in \mathbb{H}_i$.

1.3 Entangled states:

Definition 5: A composite quantum system that cannot be written as a productos state is said to be entangled.

$$|\Psi\rangle \neq \bigotimes_{i=1}^{N} |\psi_i\rangle$$

Examples

Let $\mathbb{H}_1 = \mathbb{H}_2$ with dimension $d = 2$. Write the state $|\Psi\rangle \in \mathbb{H}_1 \otimes \mathbb{H}_2$ in tensor product form

1. $|\Psi\rangle = \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right) \; = \frac{1}{2}(|0\rangle_A + |1\rangle_A) \otimes (|0\rangle_B + |1\rangle_B)$

2. $|\Psi\rangle = \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|01\rangle$

3. $|\Psi\rangle = \sqrt{\frac{1}{6}}|00\rangle + \sqrt{\frac{1}{3}}\, e^{i\frac{\pi}{3}}|01\rangle + \sqrt{\frac{1}{6}}\, e^{i\frac{\pi}{4}}|10\rangle + \sqrt{\frac{1}{3}} e^{i\frac{7\pi}{4}}|11\rangle$

4. $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + e^{i\phi}|11\rangle\right)$ (IMPOSSIBLE !)

Entanglement: q. correlations

- Entanglemet deals with a generic form of quanum correlations, and is linked to the tensorial structure of the Hilbert space.

- Entanglement is a propertie of composite quantum systems. We shall consider from now generically bipartite quantum states (Alice & Bob)

$$|\psi\rangle_{AB} \in \mathbb{H}_A \otimes \mathbb{H}_B$$

(Otfried Ghüne will tell us about multipartite quantum systems)

- Entanglemet is arguably the most genuine property of quantum physics as allows to perform tasks that otherwise are impossible.

- Entanglement is considered to be a resource for quantum information tasks. There are other resources as for instance coherence, locality, asymmetry, etc..

Bipartite Entanglement

Theorem 4 [Schmidt Decomposition]: Let $|\psi\rangle \in \mathbb{H}_A \otimes \mathbb{H}_B$. Then, there exist two orthonormal basis $\{|v_i\rangle\}_{i=1}^{d_1} \in \mathbb{H}_A, \quad \{|u_i\rangle\}_{i=1}^{d_2} \in \mathbb{H}_B$ such that

$$|\psi\rangle = \sum_{i=1}^{r \leq \min(d_1, d_2)} \lambda_i \, |v_i, u_i\rangle$$

where $\lambda_i \geq 0, \ \sum_{i=1}^{r} \lambda_i^2 = 1$ are called the Schmidt coefficients of the state.

Remark 1: The number of non-zero Schmidt coefficients of the state is called the Schmidt rank r of the state, whereas the basis $\{|v_i\rangle\}_{i=1}^{d_1} \in \mathbb{H}_A, \quad \{|u_i\rangle\}_{i=1}^{d_2} \in \mathbb{H}_B$ is known as the Schmidt basis of the state.

Remark 2: A bipartite state is a product state iff has Schmidt rank = 1, otherwise it is entangled.

Remark 3: The Schmidt decomposition is nothing else than the Singular Value Decomposition: Given a not square matrix A = U D V where D is diagonal and U and V are unitary.

Reduced states of composite systems

Consider two parties—Alice and Bob—each of which hold part of a composite quantum system in some state $|\Psi\rangle_{AB} \in \mathbb{H}_{AB} = \mathbb{H}_A \otimes \mathbb{H}_B$, where $|\mathbb{H}_A| = d_A, |\mathbb{H}_B| = d_B$,

How should Alice (Bob) describe the state of their respective system?

Clearly if $|\Psi\rangle_{AB} = |\phi\rangle_A \otimes |\chi\rangle_B$ where $|\phi\rangle_A \in \mathbb{H}_A, \quad |\chi\rangle_B \in \mathbb{H}_B$ then everything is OK!

What about entangled states?

$$|\psi\rangle = \sum_{i=1}^{r \leq \min(d_A, d_B)} \lambda_i |v_i, u_i\rangle$$

Reduced states of composite systems

Consider the bipartite state

$$|\Psi\rangle_{AB} = \sqrt{\frac{1}{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right)$$

Suppose B measures in the standard basis. What is the probability that B obtains outcome 0 or 1?

P3

$${}_{AB}\langle\Psi|\left(\mathbb{I}_A \otimes |0\rangle_B\langle 0|\right)|\Psi\rangle_{AB} = \frac{1}{2} \quad \Rightarrow \quad |\Phi\rangle_{AB} = |0\rangle_A|0\rangle_B$$

$${}_{AB}\langle\Psi|\left(\mathbb{I}_A \otimes |1\rangle_B\langle 1|\right)|\Psi\rangle_{AB} = \frac{1}{2} \quad \Rightarrow \quad |\Phi\rangle_{AB} = |1\rangle_A|1\rangle_B$$

Now suppose that B doesn't tell A the outcome of the measurement. All A can say is that her system is equally likely to be in either state!

Reduced states of composite systems

Given a bipartite pure state $|\Psi\rangle_{AB}$, the description of each subsystem is given by its reduced density matrix:

$$\rho_A \equiv Tr_B(|\Psi\rangle_{AB}\langle\Psi|) = \sum_{i_2=1}^{d_B} \langle i_2|(|\Psi\rangle_{AB}\langle\Psi|)|i_2\rangle$$

$$\rho_B \equiv Tr_A(|\Psi\rangle_{AB}\langle\Psi|) = \sum_{i_1=1}^{d_A} \langle i_1|(|\Psi\rangle_{AB}\langle\Psi|)|i_1\rangle$$

$$|\Psi\rangle_{AB} = \sqrt{\frac{1}{2}}\left(|0\rangle_A|0\rangle_B + |1\rangle_A|1\rangle_B\right) \longrightarrow$$

$$\rho_A = \frac{1}{2}(|0\rangle_A\langle 0| + |1\rangle_A\langle 1|)$$

$$\rho_B = \frac{1}{2}(|0\rangle_B\langle 0| + |1\rangle_B\langle 1|)$$

Reduced states of composite systems

Given a bipartite pure state $|\Psi\rangle_{AB}$, its Schmidt decompostion

$$|\psi\rangle_{AB} = \sum_{i=1}^{r \leq \min(d_A, d_B)} \lambda_i \, |v_i, u_i\rangle$$

gives us information about the entanglement content of the state!

Remarks 1: The Schmidt rank r cannot exceed min (d_A, d_B) since not more degrees of freedoms than the min of d_A and d_B, can be entangled between both systems.

Remark 2: A maximally entangled state has maximal Schmidt rank and all its Schmidt coeficients are equal $\lambda_i = \frac{1}{\sqrt{d}}$. Example $|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)$ $(d_1 = d_2 = 2)$

Remark 3: The Schmidt decomposition (SVD) only exist for BIPARTITE systems

Reduced states of composite systems

Given a bipartite pure state $|\Psi\rangle_{AB}$, to find its Schmidt decompostion we should: (i) calculate the reduced density matrices of the subsystems

(ii) diagonalize them.

In the Schmidt basis, both reduced density matrices are diagonal (This is the singular value decomposition!)

$$|\psi\rangle_{AB} = \sum_{i=1}^{\min(d_1,d_2)} \lambda_i |v_i, u_i\rangle$$

Since

$$\rho_A \equiv Tr_B(|\psi\rangle_{AB}\langle\psi|) = \sum_i^{d_1} \lambda_i^2 |v_i\rangle\langle v_i|$$

$$\rho_B \equiv Tr_A(|\psi\rangle_{AB}\langle\psi|) = \sum_i^{d_2} \lambda_i^2 |u_i\rangle\langle u_i|$$

1.4 Entanglement based Protocols: super-dense coding

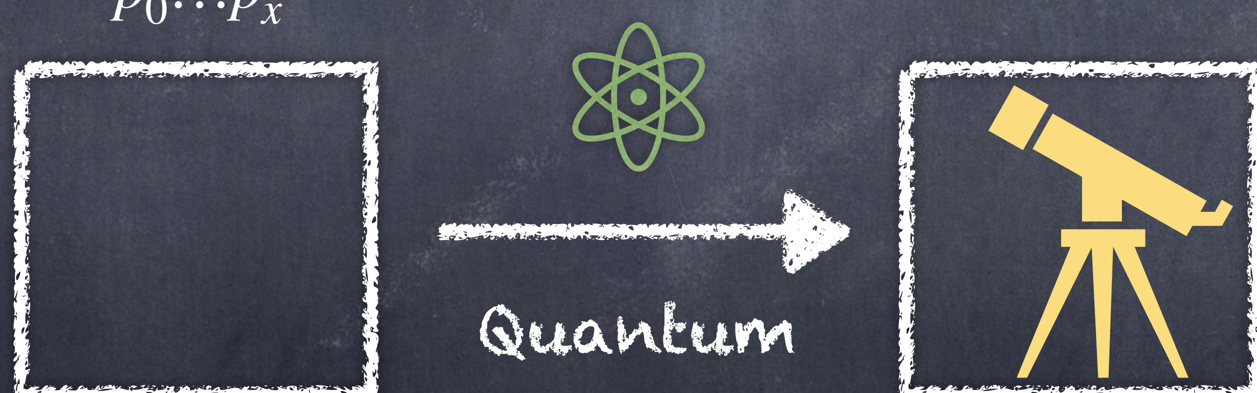
Theorem 2: Holevo bound: n-qubits **cannot** carry more information (classical) than n bits (very important theorem)

Alice

$\rho_x \quad x = 0, \ldots n$

$p_0 \cdots p_x$

Bob



Quantum channel

1.4 Protocols: super-dense coding

Super-Dense Coding: Alice wants to send two bits of information (classical) to Bob with a single use of a channel.
How? Sharing forhand a maximally entangled state !
Alice has bit a=(0,1) and the bit b=(0,1) and shares a maximally entangled state of two qubits of the form:

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

:

ALICE want sto send: she does and sends her qubit to Bob BOB measures

$$00: \quad do\ \ nothing\ \ |\Phi^+\rangle_{AB} \longrightarrow |\Phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$01: \quad do\ \ Xrotation\ \ |\Phi^+\rangle_{AB} \longrightarrow |\Phi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$10: \quad do\ \ NOT\ \ Z\ \ |\Phi^+\rangle_{AB} \longrightarrow |\Psi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$11: \quad do\ \ iYrotation\ \ |\Phi^+\rangle_{AB} \longrightarrow |\Psi^-_{AB}\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$$

Example of the use of pure state entanglement: super-dense coding

PROTOCOL example:

$$|\Phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

1) if a=1 (b=1) apply a σ_z (σ_x) to the qubit A of the state $|\Phi^+\rangle_{AB}$.

(2) Send qubit A of $|\psi\rangle_{AB}$ to Bob

(3) Bob performs a CNOT gate $CNOT = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$

Example of the use of pure state entanglement: superdense coding

(4) Bob performs a Hadamar gate on control target $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

(5) Bob measures on his qubits to extract the value of the 2 bits.

Let's do it:

(i) write the protocol as a quantum circuit

(ii) classical bits are used here a controled bits. Depending on their value Alice doe
one operation or another.

(iii) For instance if Alice wants to send (0,0), the protocol gives the following output

$$|\Phi^+\rangle_{AB} \Rightarrow_{P1} |\Phi^+\rangle_{AB} \Rightarrow_{P3} \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)|0\rangle \Rightarrow_{P4} |00\rangle$$

What we saw yesterday...

Definition 1: A composite pure quantum system is said to be in a product state iff

$$|\Psi\rangle = \bigotimes_{i=1}^{N} |\psi_i\rangle$$

where $|\Psi\rangle \in \mathbb{H}_{\text{Total}} = \bigotimes \mathbb{H}_i$ and $|\psi_i\rangle \in \mathbb{H}_i$. Otherwise it is ENTANGLED

Definition 2: Given a bipartite pure state $|\Psi\rangle_{AB}$, it can always be written in its Schmidt decompostion

$$|\psi\rangle_{AB} = \sum_{i=1}^{r \leq \min(d_A, d_B)} \lambda_i |v_i, u_i\rangle$$

gives us information about the entanglement content of the state!

1.5 Mixed states: Ensembles of quantum states

Definition: An ensemble of pure states (mixed state) describes a situation where a quantum system can be in any one of a different pure states $|\psi_i\rangle \in \mathbb{H}$ with probability p_i.

$$\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

Remark 1: It is customary to represent a particular ensemble of quantum states as

$$\{p_i, |\psi_i\rangle\} \longrightarrow \rho = \sum_i p_i |\psi_i\rangle\langle\psi_i|$$

where $p_i > 0$, $\sum p_i = 1$, that is, a convex combination of projectors onto pure states.

Remark 2: To each ensemble we can associate a density matrix but to each density matrix we can associate many different ensambles !!

Recap: The Postulates of Q.M

in the most general terms possible...

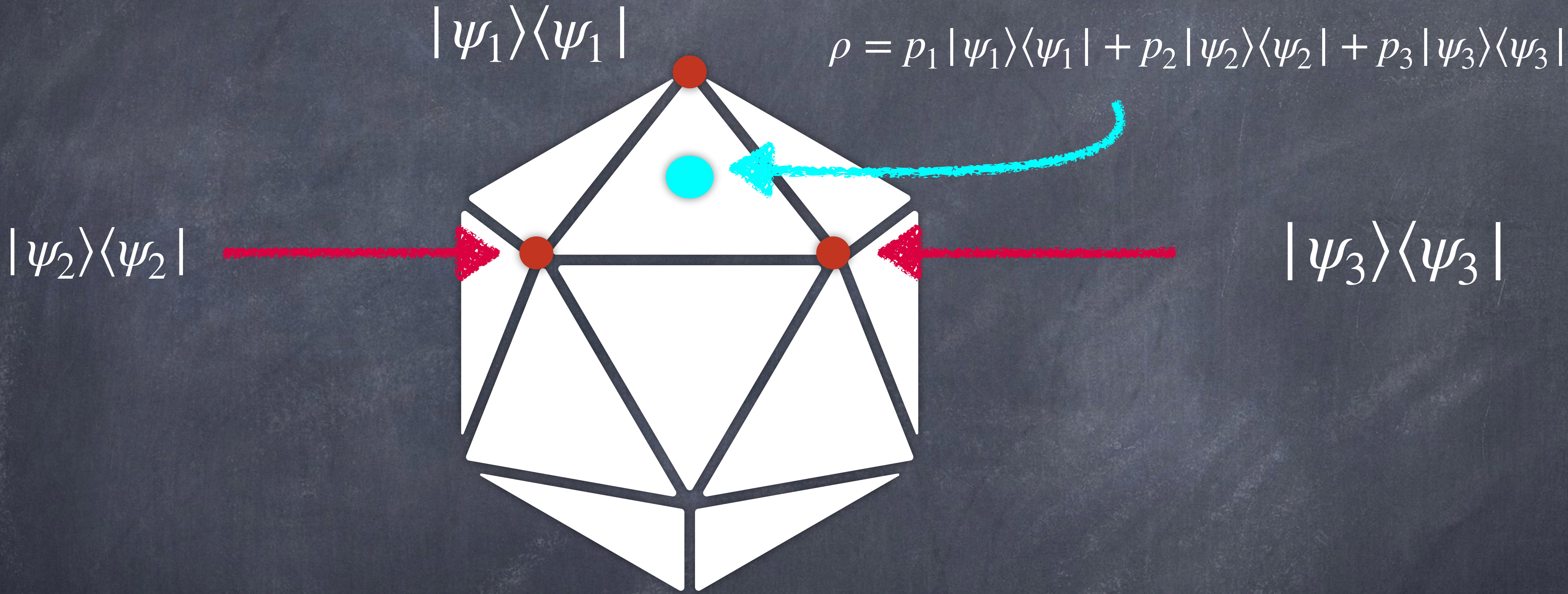
Postulate 1: Associated to any physical system is a density operator $\rho \in \mathscr{B}(\mathbb{H}), \rho > 0, \mathrm{tr}(\rho) = 1$. If the system is known to be in state ρ_i with probability p_i then $\rho = \sum_i p_i \rho_i$.

Postulate 2: The evolution of a quantum system is described by a completely positive, (generally time-dependent) trace non-increasing map $\mathscr{E} : \mathscr{B}(\mathbb{H}_{\mathrm{in}}) \to \mathscr{B}(\mathbb{H}_{\mathrm{out}})$ such that

$$\rho(t) = \mathscr{E}_t\left(\rho\right)$$

Postulate 4: The state of a composite quantum system is described by a density operator $\rho \in \mathscr{B}\left(\bigotimes_{i=1}^{N} \mathbb{H}_i\right)$. If the state of each constituent system is given by ρ_i then the state of the composite system is $\rho = \bigotimes_{i=1}^{N} \rho_i$

The "world" according to Q.I.T: convex set!



Extremal points: projectors on pure states

facets: some density matrix rank deficient

inside: density matrices

Composite systems in QIT: convex sets and convex polytops

Correlations, even classical ones, mean

$$p(ab|xy) \neq p(a|x)p(y,b)$$

The world according to QIT: convex sets and convex polytops

1. NS (Non-Signaling)

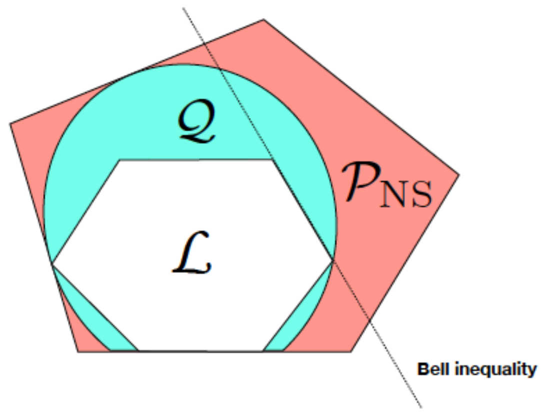
$$\mathscr{P}_{NS} \rightarrow p(a|x) = p(a|xy) = \Sigma_b p(ab|xy)$$
$$p(b|y) = p(b|xy) = \Sigma_a p(ab|xy)$$

2. Local

$$\mathscr{L} \rightarrow p(ab|xy) = \int_\Lambda d\lambda \, p(a|x\lambda) \, p(b|y\lambda)$$

3. Quantum

$$\mathscr{Q} \rightarrow p(ab|xy) = Tr(\rho_{AB}[M_{a|x} \otimes M_{b|y}])$$



$$\mathscr{L} \subset \mathscr{Q} \subset \mathscr{P}_{NS}$$

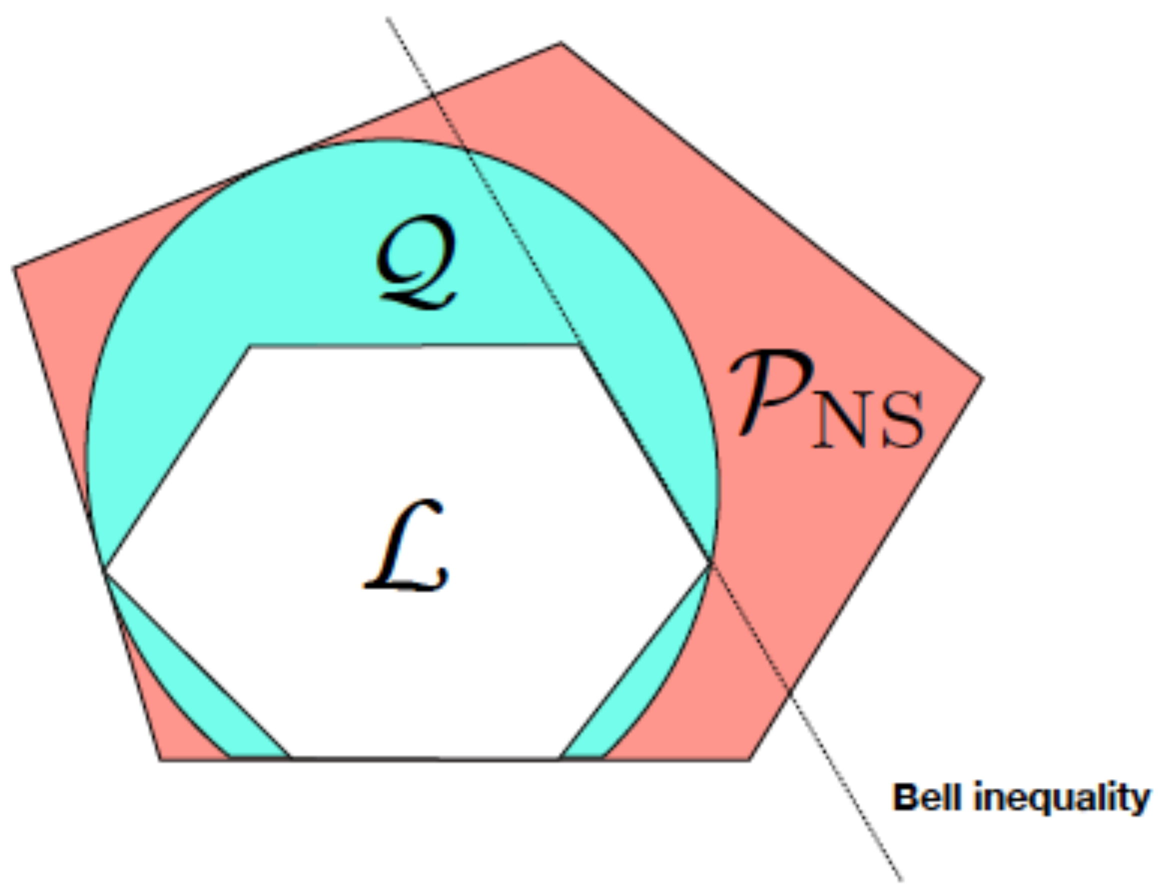
A Bell inequality , is a linear inequality for the probabilities p (ab|xy) that is necessarily verified by any model satisfying the locality condition

Quantum

$$\mathcal{Q} \rightarrow p(ab|xy) = Tr(\rho_{AB}[M_{a|x} \otimes M_{b|y}]) \Leftrightarrow$$

Convex set whose extremal points

$|\Psi\rangle\langle\Psi|$ $\quad |\Psi\rangle \in \mathscr{H}$



$\rho \in \mathscr{B}(\mathscr{H});\ \rho > 0;\ Tr(\rho) = 1$

$\rho = \Sigma_i\, p_i |\Psi_i\rangle\langle\Psi_i|$

The quantum world

$$\mathscr{L} \subset \mathcal{Q}$$

Horodecki ^3 Rev. Mod. Phys (2009)

Entanglement in mixed states

Definition: A bipartite quantum state $\rho_{AB} \in \mathscr{B}(\mathbb{H}_A \otimes \mathbb{H}_B)$ is said to be separable if it the can be written as

$$\rho_{AB} = \sum_i p_i\,(\rho_i^A \otimes \rho_i^B) = \sum_i q_i\,(|e_i\rangle_A\langle e_i| \otimes |f_i\rangle_B\langle f_i|)$$

with $p_i \geq 0$ and $\sum p_i = 1$, ($q_i \geq 0$ and $\sum q_i = 1$). In other words the state ρ_{AB} is separable iff it is a convex combination of projectors on product states (or projectors on local states).

Remarks: To be separable means that the state can be prepared using local operations and classical communication. Such operatos are called LOCC

Entanglement for bipartite pure states

The following definitions are equivalent

1. $$|\Psi\rangle_{AB} \neq |\psi\rangle_A \otimes |\phi\rangle_B$$

2. The Schmidt decomposition of $|\Psi\rangle_{AB}$ rank r>1

$$|\psi\rangle_{AB} = \sum_{i=1}^{r \leq \min(d_A, d_B)} \lambda_i \, |v_i, u_i\rangle$$

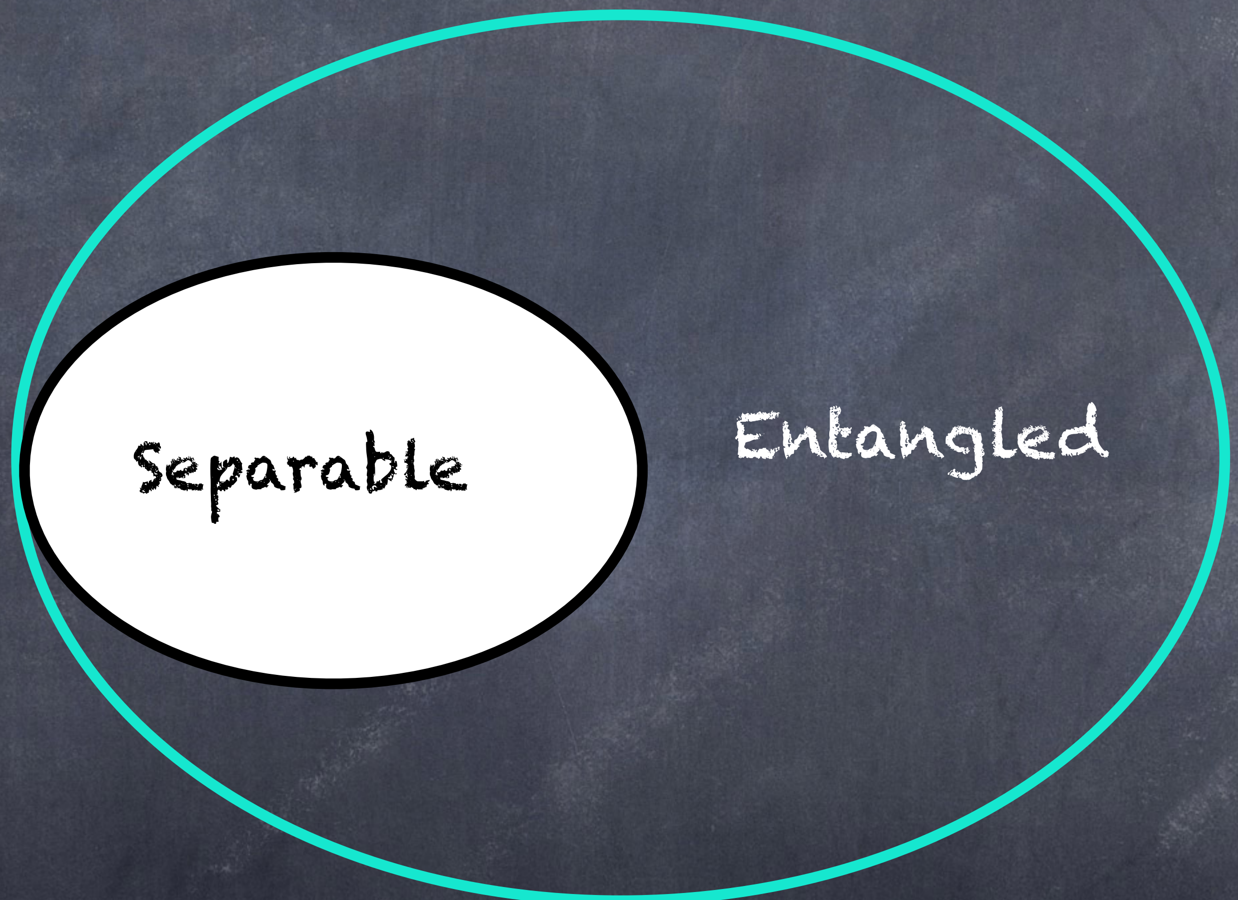
3. $|\Psi\rangle_{AB}$ violates a Bell inequality

Lecture 2

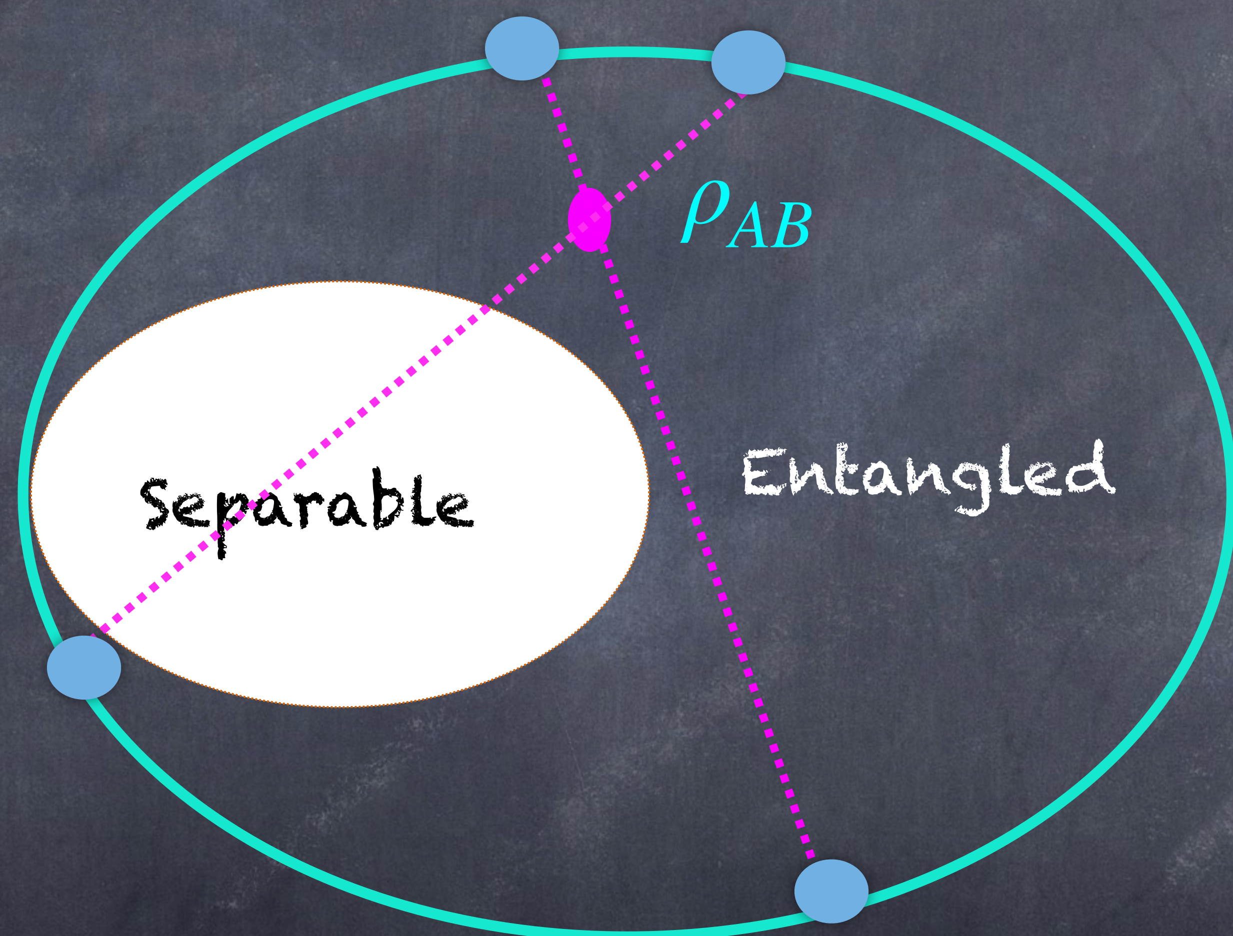
2.1 Quantification of entanglement

- Entanglement permits to do tasks that cannot be done with classical states: superdense coding, teleportation, and other algorithms
- Entanglement is therefore a RESOURCE for quantum information. Free states are separable states and LOCC are free operatins.
- Unit of entanglement is the e-bit, that is, the entanglement contained in a maximally entangled bipartite state of two-qubits
- What is the entanglement in an arbitrary pure state $|\Phi_{AB}\rangle$?
- What is the amount of entanglement in a mixed state ρ_{AB} ?

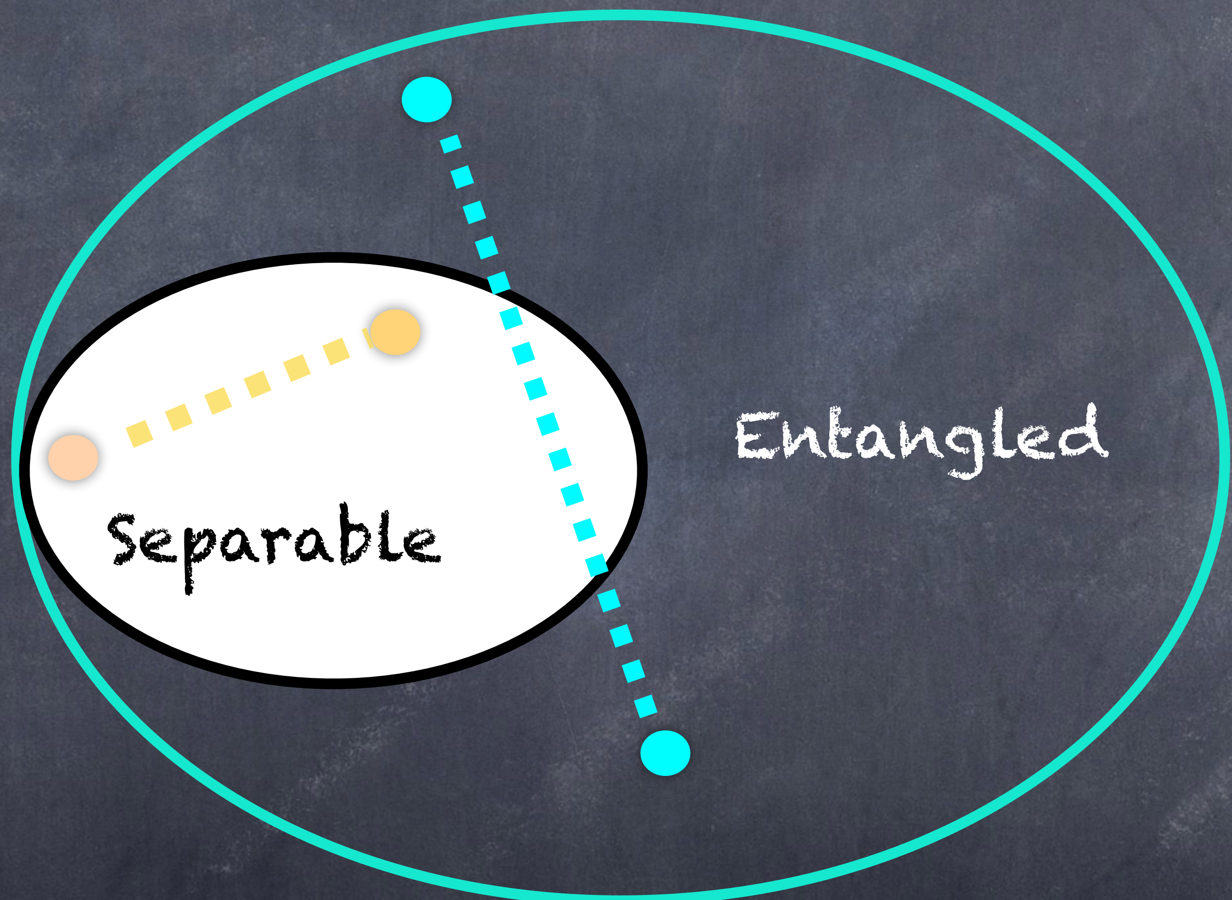
The set of composite quantum systems



The set of composite quantum systems

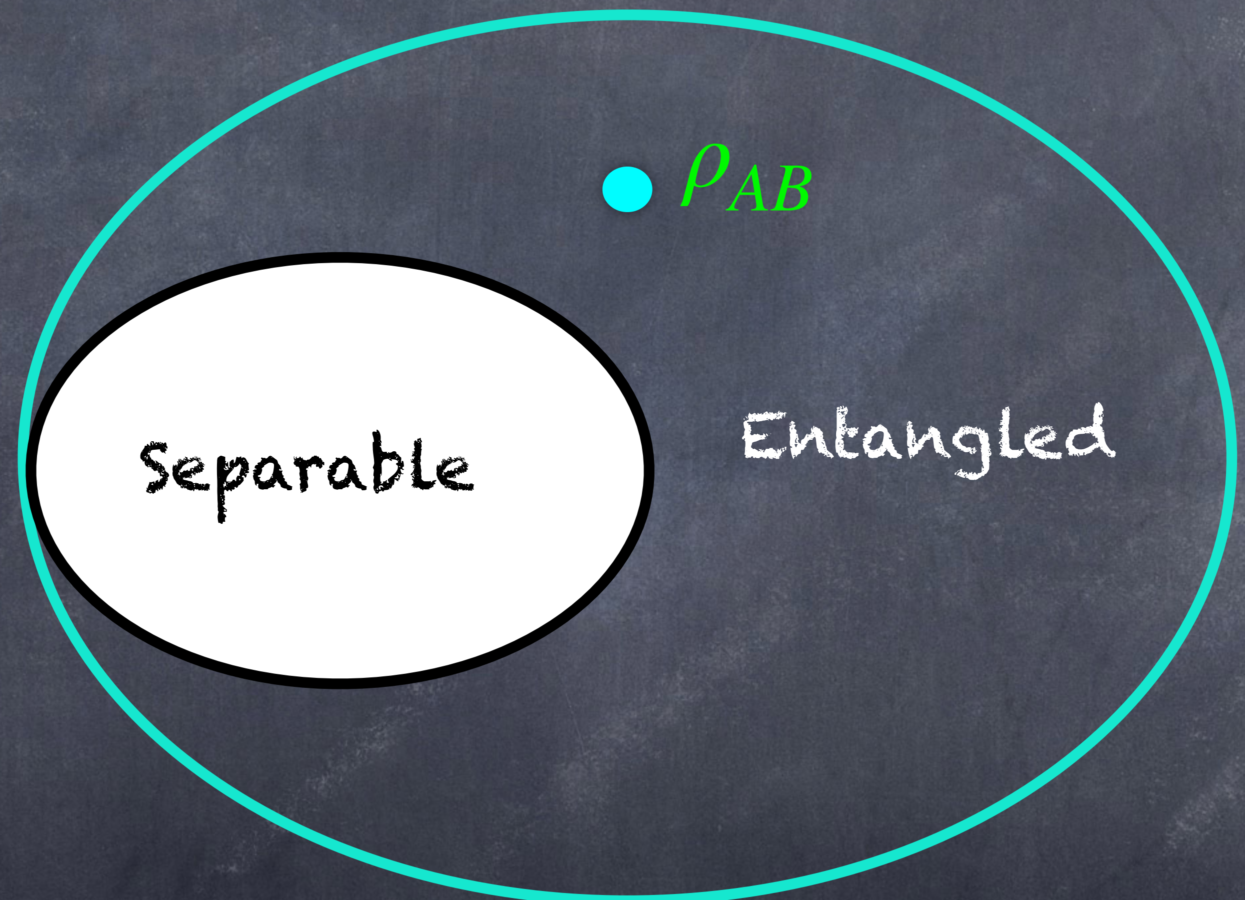


The set of composite quantum systems



- $\rho_{AB} = p_1(\rho_A^1 \otimes \rho_B^1) + p_2(\rho_A^1 \otimes \rho_B^2)$
- $\tilde{\rho}_{AB} = q_1(\tilde{\rho}_A^1 \otimes \tilde{\rho}_B^1) + q_2(\tilde{\rho}_A^2 \otimes \tilde{\rho}_B^2) + q_3(\tilde{\rho}_A^3 \otimes \tilde{\rho}_B^3)$

The set of composite quantum systems



How much entangled ?

Entanglement Measures

- A measure of entanglement E must fullfill:

1. $E(\rho) \geq 0$ for $\forall \rho \in \mathscr{B}(\mathbb{H}_A \otimes \mathbb{H}_B)$
2. $E(\sigma_{AB}) = 0$ if $\sigma_{AB} = \sum_i p_i \sigma_i^A \otimes \sigma_i^B$, that is, if the state is separable
3. $E(U_A \otimes U_B \rho U_A^\dagger \otimes U_B^\dagger) \leq E(\rho)$
4. Given a LOCC map Λ, $E(\Lambda(\rho)) \leq E(\rho)$
5. (*) Convexity: it may happen that $E(\sum p_i \rho_i) \leq \sum p_i E(\rho_i)$
6. (*) Additivity $E(\rho^{\otimes n}) = nE(\rho)$

- Remarks: (i) Convexity and Additivity are not necessary !
- (ii) There are many different entanglement measures and normally they are not equivalent!

2.2 Entanglement of pure states

Definition 2.1: The entanglement entropy is the standard entanglement measure used for bipartite pure state $|\psi\rangle_{AB}$

$$E(|\psi\rangle_{AB}) = S(\rho_A) = S(\rho_B)$$

where $S(\rho) = -Tr\rho\log(\rho)$ is the von Neumann entropy and

$\rho_A(\rho_B)$ are the reduced density matrices (marginals), i.e.
$\rho_A = Tr_B(|\Psi\rangle_{AB}\langle\Psi|)$

There are two measures of bipartite entanglement conceptually very important which lead to the definition of entanglement entropy. The latter isthe unique measure of bipartite entanglement for pure states which is operationally meaningful.

Entanglement of pure states

Remarks:

- if $|\psi\rangle_{AB} = |\Phi_A\rangle \otimes |\varphi_B\rangle \Rightarrow E(|\psi\rangle_{AB}) = 0$ (product states have zero entanglement)

- if $|\Psi\rangle_{AB} = \sum_{i=1}^{M} \sqrt{\lambda_i}\,|e_i\rangle|f_i\rangle \Rightarrow E(|\Psi\rangle_{AB}) = -\sum \lambda_i \log \lambda_i$ (Shannon entropy)

- if $|\psi\rangle_{AB} = |\Psi^-\rangle_{AB} = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \Rightarrow E(|\Psi^-\rangle_{AB}) = 1$ (an e-bit) SINGLET

- if $|\Psi^+\rangle_{AB} = \frac{1}{\sqrt{d}} \sum_{i=1}^{d} |i\rangle|i\rangle \Rightarrow E(\Psi^+\rangle_{AB}) = \log_2 d$

- If the pure state is N-multipartite $|\Psi\rangle_{1,2,..N}$ we can always calculate the entanglement entropy of a given bipartite splitting, i.e. $E(|\Psi\rangle_{AB})$ where AB is any bipartite splitting of the N parties

2.3 Entanglement of mixed states

Recall: To every ensemble of quantum states $\{p_i, |\psi_i\rangle\}$ one can associate a density operator $\rho = \sum_i p_i |\psi_i\rangle\langle\psi_i| \in \mathscr{B}(\mathbb{H})$.

Entanglement measures: convex roof extensions!

Entanglement of Formation E_{oF}

Definition 2.1: Given a bipartite mixed state ρ_{AB}, the entanglement of formation is defined as:

$$E_F(\rho_{AB}) = \min_{\{p_i, |\psi^i\rangle_{AB}\}} \sum p_i E(|\psi^i\rangle_{AB})$$

Remarks: (i) The infimum is taken over all possible ensembles compatibles with the mixed state

(ii) Meaning: The entanglement of formation tell us on average how many entanglement is need to form the state

Entanglement of mixed states

Entanglement of Formation E_{oF}

$$E_F(\rho_{AB}) = \min_{\{p_i, |\psi^i\rangle_{AB}\}} \sum p_i E(|\psi^i\rangle_{AB})$$

The convex roof optimization is VERY HARD to do, but for 2-qubit mixed states it can be computed via the concurrence.

Definition: The concurrence of a 2-qubit pure state $|\psi\rangle_{AB}$ is a measure of entanglement given by

$$C(|\psi\rangle_{AB}) = |\langle\psi_{AB}|\tilde{\psi}_{AB}\rangle| \text{ where } |\tilde{\psi}\rangle_{\mathrm{AB}} = \sigma_{\mathrm{y}} \otimes \sigma_{\mathrm{y}} |\psi\rangle^*_{\mathrm{AB}}$$

using the computational basis {|00>,|01>,|10>,|11>}

Entanglement of mixed states

Definition: The concurrence of a 2-qubit mixed state ρ_{AB} is a measure of entanglement given by

$$C(\rho_{AB}) = min(0,\lambda_1 - \lambda_2 - \lambda_3 - \lambda_4)$$

where λ_i are the eigenvalues in decreasing order of the operator $R = \sqrt{\sqrt{\rho_{AB}}\tilde{\rho}_{AB}\sqrt{\rho_{AB}}}$ where $\tilde{\rho}_{AB} = (\sigma_y \otimes \sigma_y)\rho^*_{AB}(\sigma_y \otimes \sigma_y)$

Theorem: The entanglement of formation of a 2-qubit mixed state ρ_{AB} is

$$E(\rho_{AB}) = F(C(\rho_{AB})) = h[\frac{1+\sqrt{1-C^2}}{2}]$$

and $h[x] = -x\log x - (1-x)\log(1-x)$

Entanglement of mixed states: entanglement cost and entanglement distillation

For bipartite systems of qubits (Hilbert space of dimension 2)

Entanglement cost and entanglement of distillation are two dual measures defined in the assymptotic limit.

1- How many singlets do I need to prepare a bipartite entangled state ρ_{AB}?

2- How many singlets can I distill from a given state ρ_{AB} if I have many copies of the state.

fr

Entanglement of mixed states:
entanglement cost and entanglement distillation

Definition: The entanglement cost of a mixed state ρ_{AB} denoted by $E_c(\rho_{AB})$ is the infimum over all sequences of LOCC protocols such that given m-copies of the singlet state $|\Psi^-\rangle_{AB}^{\otimes m}$

$$|\Psi^-\rangle_{AB}^{\otimes m} \underset{L\in LOCC}{\rightarrow} \sigma \text{ such that } \mathrm{D}(\rho_{\mathrm{AB}}^{\otimes \mathrm{n}}, \sigma) \underset{\mathrm{n}\rightarrow\infty}{\rightarrow} 0$$ where D is a proper distance.

The entanglement cost of ρ_{AB} is defined as

$$E_c(\rho_{AB}) = \underset{L\in LOCC}{min} \; (\lim_{n\rightarrow\infty} \frac{m}{n})$$

$$E_c(\rho_{AB}) = \lim_{n\rightarrow\infty} \frac{E_F(\rho_{AB}^{\otimes n})}{n}$$

in simple words if defines the number e -bits one needs to create a entangled state σ which is the closest to the one we could achieved if we had n copies of our state using only LOCC operations. fr

Entanglement of mixed states: entanglement cost and entanglement distillation

Definition: The entanglement of destillation of a mixed state ρ_{AB} denoted by $E_D(\rho_{AB})$ is the suprem over all sequences of LOCC protocols L such that given n-copies of our state $\rho_{AB}^{\otimes n}$ we approach a state whose distance to $|\Psi^-\rangle_{AB}^{\otimes m}$ singlets is zero in the assymptotic limit.

If this is not possible $E_D = 0$. The entanglement of distillation is the supremum over all possible destillation rates.

The entanglement distillation of ρ_{AB} is defined as

$$E_D(\rho_{AB}) = \max_{L \in LOCC} \left(\lim_{n \to \infty} \frac{m}{n} \right)$$

where $D(|\Psi^-\rangle^{\otimes m}, \sigma_n) \underset{n \to \infty}{\rightarrow} 0$

Entanglement cost and entanglement distillation

Theorem The entanglement of destillation is always smaller equal to the entanglement cost

$$E_D(\rho_{AB}) \leq E_c(\rho_{AB})$$

Theorem. Any other measure of entanglement fullfills

$$E_D(\rho_{AB}) \leq E(\rho_{AB}) \leq E_c(\rho_{AB})$$

However, for pure states and only for them,
the entanglement cost and the entanglement of distillation coincide and are given by the von Neumann entropy of the subsystems

$$E(|\psi\rangle_{AB}) = S(\rho_A) = S(\rho_B)$$

Negativity

Definition: The negativity of a composite quantum systems ρ_{AB} is the absolute sum of the negative eigenvalues of the partial transpose density matrix

$$\mathcal{N}(\rho_{AB}) = \frac{||\rho_{AB}^{T_B}|| - 1}{2} \text{ where } ||A|| = Tr(\sqrt{A^\dagger A})$$

Partial transposition

Definition: Let ρ_{AB} be a bipartite density matrix that can be expressed as

$$\rho_{AB} = \sum_{\substack{1 \leq i,j \leq d_A \\ 1 \leq \mu,\nu \leq d_B}} \rho_{ij}^{\mu\nu} (|i\rangle\langle j|)_A \otimes |\mu\rangle\langle\nu|_B)$$

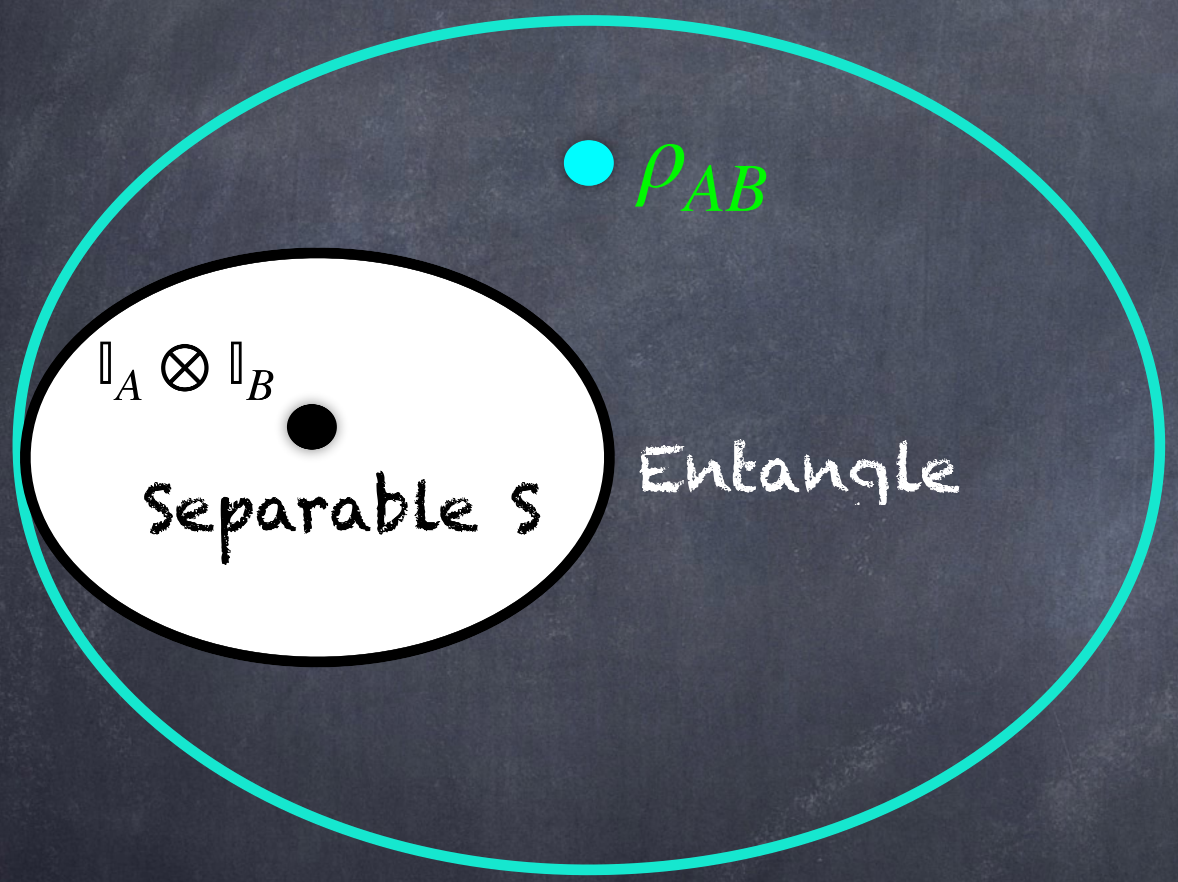
the partial transpose of the density matrix ρ_{AB} with respect to system A is

$$\rho_{AB}^{T_A} = \sum_{\substack{1 \leq i,j \leq d_A \\ 1 \leq \mu,\nu \leq d_B}} \rho_{ij}^{\mu\nu} (|j\rangle\langle i|)_A \otimes |\mu\rangle\langle\nu|_B)$$

A similar definition exist for the partial transpose w.r.t subsystem B

Entanglement measures
robustness and BSA

$$\mathbb{I}_A = \frac{1}{2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$$



Robustness

$$R(\rho_{AB}) = min_{\sigma \in S, s \in R}\left(\frac{\rho_{AB} + s\sigma}{1+s} \in S\right)$$

BSA: Best separable approximation

$$\rho_{AB=} = \lambda \sigma_S + (1-\lambda)\,|\Psi\rangle_{AB}\langle\Psi|$$

How much entangled ?

Theorem : Any other entangled measure is between these two.

Entanglement Criteria

Theorem: PPT criterion. If a state ρ_{AB} is separable, then $\rho_{AB}^{T_A} \geq 0$ and $\rho_{AB}^{T_B} = (\rho_{AB}^{T_A})^T \geq 0$

Proof: Trivial applying partial transposition on a separable state. A state that fullfills their partial transposes are positive is called a PPT (positive partial transpose) state.

Recall: $\rho_{AB}^{T_A} \geq 0$ means its eigenvalues are all larger or equal zero.

Theorem: If $dim(\mathbb{H}_A) \times dim(\mathbb{H}_B) \leq 6$, PPT is sufficient and necessary to proof the state is separable.

In higher dimensions, PPT criterion is NECESSARY for separability but not SUFFICIENT, meaning that there are states that are entangled and fulfill that $\rho_{AB}^{T_A} \geq 0$ and $\rho_{\mathrm{AB}}^{\mathrm{T_B}} \geq 0$.

Entanglement Criteria

Theorem: Entropy entanglement criterion. If a state ρ_{AB} is separable, then

$$S(\rho_{AB}) \geq S(\rho_A) \text{ and } \mathrm{S}(\rho_{\mathrm{AB}}) \geq \mathrm{S}(\rho_{\mathrm{B}})$$

where $S(\rho) = -Tr(\rho \log \rho)$ is the von Neumann entropy of the state.

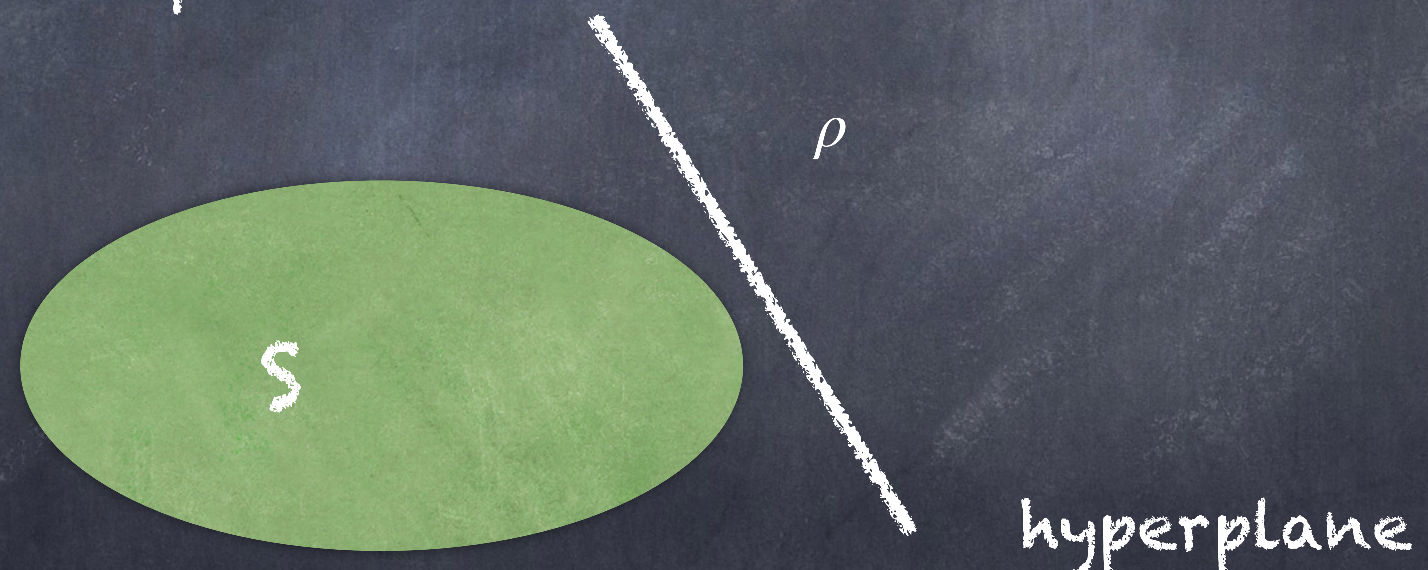
From all operationa entanglement criteria, PPT is probably the strongest but there are entangled states that are detected by the majorization or by entropy criterion that are not detected by PPT.

Non operational Entanglement Criteria

There are entanglement criteria that depend on the state we consider, for that reason they are called non-operational criteria

Lemma: $Tr(\rho_{AB}^{T_A}\sigma_{AB}) = Tr(\rho_{AB}\sigma_{AB}^{T_A})$

Theorem: Han-Banach theorem. Let S be a convex compatct set in a finite dimensional Banach space. Let ρ be a point with $\rho \notin S$ then there exist a hyperplane that separates ρ from S

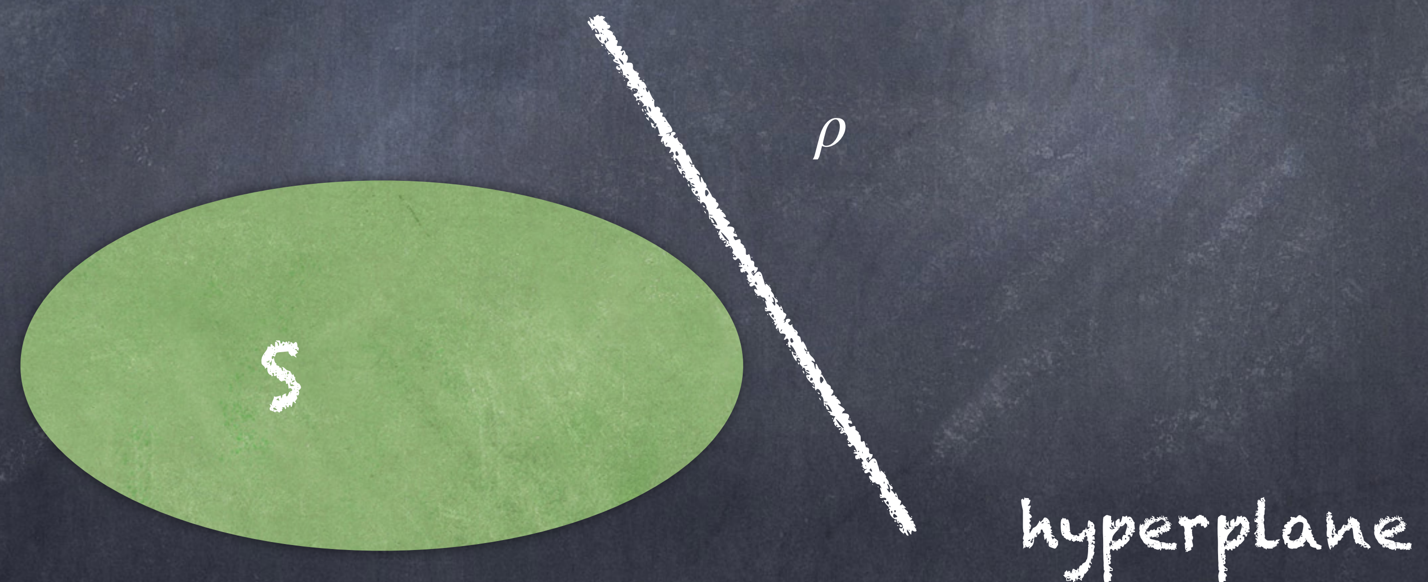


Entanglement witness

Definition: An Hermitian operators (observable) W is called an entanglement witness (EW) if and only if

1. $Tr(W\rho_S) \geq 0 \;\; \forall \rho \in S$ where S is the set of separable states

2. There exist at least one entangled state ρ such that $Tr(W\rho) < 0$



Entanglement witness

Definition: An entanglement witness is called decomposable if and only if there exsist operators P and Q such that

$$W = P + Q^{T_A} \text{ with } \mathrm{P}, \mathrm{Q} \geq 0$$

Lemma: A decomposable entanglement witness cannot detect PPT entangled states

Theorem:

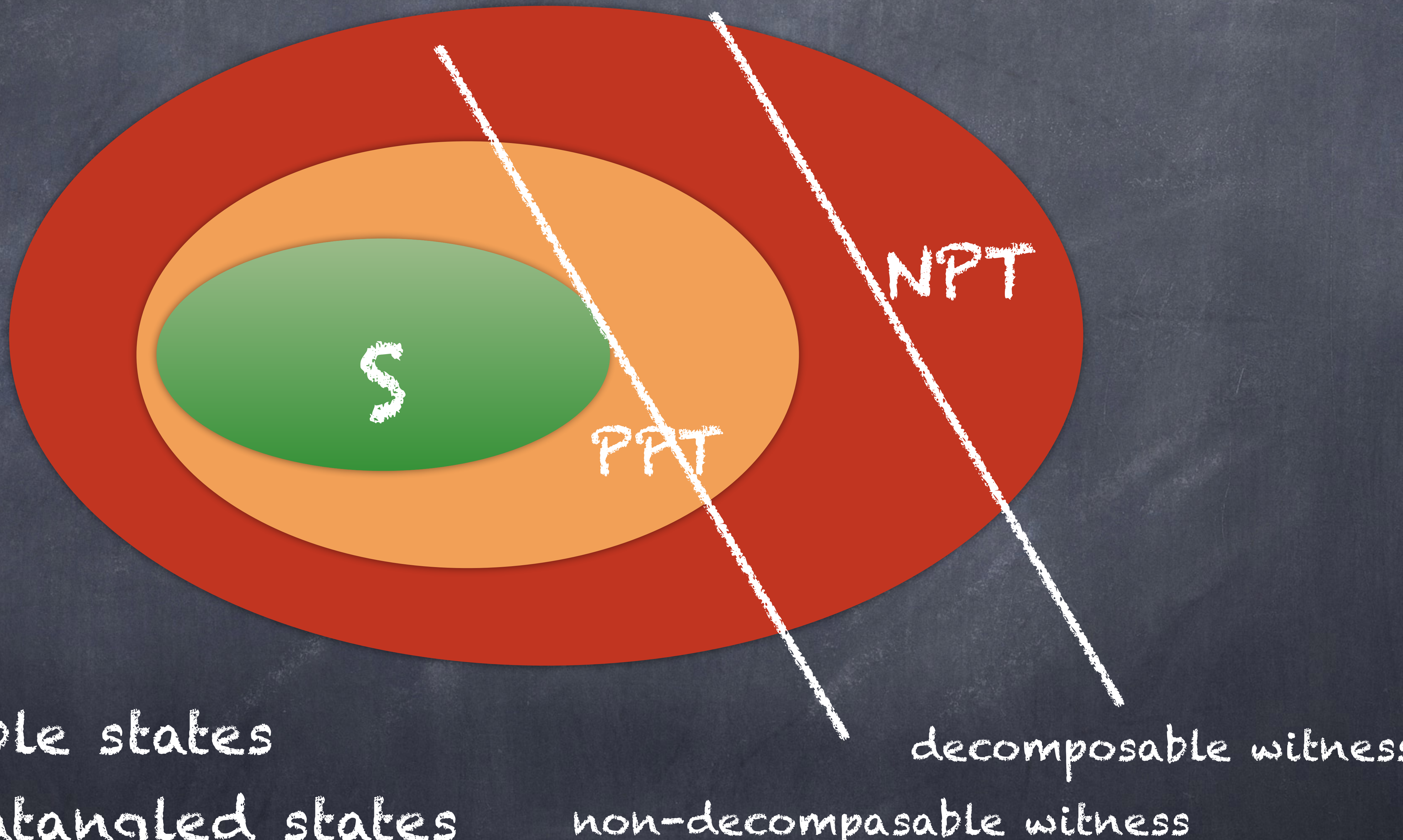
1. ρ is entangled if and only if there exist a witness W that detects it: $Tr(W\rho) < 0$.

2. ρ is an entangled PPT state if and only if there exist a non decomposable entanglement witness that detects it

3. σ is a separable state if and only if $Tr(W\sigma) \geq 0$ for all entanglement witnesses.

Entanglement witness

The structure of the space of quantum states



S sepable states

PPT entangled states

NPT entangled states

Entanglement witness

Example: Let us construct a witness for a bipartite pure maximally entangled state. We take $|\Phi^+\rangle = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

A witness operator is immediate construct as $W = Q^{T_A} = (|\Phi^+\rangle\langle\Phi^+|)^{T_A}$

$$Q = \begin{pmatrix} 1/2 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 1 & 1/2 \end{pmatrix} \qquad Q^{T_A} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{pmatrix} = (1 - 2|\Psi^-\rangle\langle\Psi^-|)$$

To show that W is a witness we need to show that

Entanglement witness

To show that $W = Q^{T_A} = (|\Phi^+\rangle\langle\Phi^+|)^{T_A}$ is a witness we need to show

(i) $\mathrm{Tr}(W\rho_{\mathrm{sep}}) \geq 0$, this is equivalent to show that $\mathrm{Tr}(W|e,f\rangle\langle e,f|) = \langle e,f|W|e,f\rangle \geq 0$. It suffices to write $|e\rangle = a_o|0\rangle + b_0|1\rangle$, and $|f\rangle = a_1|0\rangle + b_1|1\rangle$, with $a_i, b_i \in \mathbb{C}$

(ii) There exist one entangled state such that $\mathrm{Tr}(W\rho_e) < 0$. Choose $\rho_e = |\Psi^-\rangle\langle\Psi^-|$. Trivially $\mathrm{Tr}(W\rho_e) = -1$

$$Q^{T_A} = \begin{pmatrix} 1/2 & 0 & 0 & 0 \\ 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1 & 1/2 \end{pmatrix} = (1 - 2|\Psi^-\rangle\langle\Psi^-|)$$